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Universal Hubbard models with arbitrary symmetry

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Abstract

We propose a general framework that leads to one-dimensional XX and Hubbard models in full generality, based on the decomposition of an arbitrary vector space (possibly infinite dimensional) into a direct sum of two subspaces, the two corresponding orthogonal projectors allowing one to define a R -matrix of a universal XX model, and then of a Hubbard model using a Shastry type construction. The QISM approach ensures integrability of the models, the properties of the obtained R -matrices leading to local Hubbard-like Hamiltonians.

In all cases, the energies, the symmetry algebras and the scattering matrices are explicitly determined. The computation of the Bethe Ansatz equations for some subsectors of the universal Hubbard theories are determined, while they are fully computed in the XX case. A perturbative calculation in the large coupling regime is also done for the universal Hubbard models.

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1 Introduction

The celebrated Hubbard model, introduced in the sixties [1, 2] in order to study strongly correlated electrons, has been widely studied since then, essentially in connection with condensed matter physics. Due to the extent of the literature on the subject, the reader is invited to refer to the books [3, 4] and references therein. The eigenfunctions and energies of the 1D-model are known by means of the Bethe ansatz thanks to the works of Lieb and Wu [5], the complete set of eigenstates being obtained in [6], exploiting the $SO(4)$ symmetry present in the one-dimensional case.

The essence of the Hubbard model is rather fascinating: although the one-dimensional model was solved in the late sixties, the understanding of the model in the light of the quantum inverse scattering method became clear only twenty years after with the works of Shastry [7, 8] and Olmedilla et al. [9]. The main idea is to couple the R -matrices of two independent XX models, through a term depending on the coupling constant of the Hubbard potential. The complete proof of the Yang–Baxter relation for the Hubbard R -matrix was given by Shiroishi and Wadati [10].

Since then, generalizations of the Hubbard model in the framework of the R -matrix formalism have been proposed. A first step was done by Maassarani [11, 12], extending the R -matrix construction to the $gl(N)$ case.

The appearance of the Hubbard model in the context of $N = 4$ super Yang–Mills theory led to new motivations to investigate further the supercase. The Hubbard model at half-filling, when treated perturbatively in the coupling [13], reproduces the long-ranged integrable spin chain of Ref. [14] as an effective theory. It was conjectured in [14] that the Hamiltonian of this chain be an all-order description of the dilatation operator of $N = 4$ super Yang–Mills in the $su(2)$ subsector. There may be the possibility that some integrable extension of the Hubbard model (e.g. involving superalgebras) could be put in relation to other subsectors of the $N = 4$ super Yang–Mills theory.

The Hubbard model has also arisen in [15], where an S -matrix for a long-range interacting integrable quantum spin chain with centrally extended $su(2|2)$ symmetry was constructed. This S -matrix was shown to be proportional to Shastry’s R -matrix up to a dressing phase. This phase indeed leads to a breakdown of the conjecture of [14] beyond three loops and to transcendental contributions to the dilatation operator eigenvalues. However, the proposal of a string Bethe ansatz and the appearance of the Hubbard R -matrix in the study of integrable structures in view of the AdS/CFT correspondence ask for learning more about generalized Hubbard models. Hopefully, the statistical mechanics community may also find interest in exploring these structures.

A superalgebraic generalization of the Hubbard model in the spirit of Shastry’s construction has been proposed by the authors in [16] and [17], where a general approach to constructing a number of super Hubbard models was developed. Each of the obtained models can be treated perturbatively and thus gives rise to an integrable long-ranged spin chain as an effective theory. The symmetry of the super Hubbard model based on $gl(\mathbf{m}|\mathbf{n})$ was shown to be $gl(\mathbf{m} - 1|\mathbf{n} - 1) \oplus gl(1|1) \oplus gl(\mathbf{m} - 1|\mathbf{n} - 1) \oplus gl(1|1)$. In this paper, we propose a general framework that leads to XX and Hubbard models in full generality. It may also constitute an interesting starting point for dealing with integrable bosonic Hubbard models. More precisely, it is based on the decomposition of an arbitrary vector space (possibly

infinite dimensional) into a direct sum of two subspaces, the two corresponding orthogonal projectors allowing one to define a R -matrix of a universal XX model, and then of a Hubbard model using a Shastry type construction. The QISM approach ensures the integrability of the models, the properties of the obtained R -matrices leading to local Hubbard-like Hamiltonians. In the finite dimensional case, they can be interpreted in terms of ‘electrons’ after a Jordan–Wigner transformation [18] (see some examples in [16]).

The plan of the paper is as follows. In section 2, we extend the construction of XX models for algebras [19] and superalgebras [16] to the case of an arbitrary vector space, possibly infinite dimensional. We focus to the general case $gl(\mathfrak{m}|\mathfrak{n})$ in section 3, in which the Hamiltonians are explicitly constructed and the Bethe ansatz equations computed. In section 4, we tackle with the case of universal Hubbard models, performing in the same way the calculation of the R -matrices, the transfer matrices and corresponding Hamiltonians. For both models (XX and Hubbard), the energies and the symmetry algebra, which is related to the choice of the projectors, are determined, and the corresponding charges computed. Section 5 is devoted to the Bethe ansatz equations for universal $gl(\mathfrak{m}|\mathfrak{n})$ Hubbard models. The computation of the scattering matrix of the universal Hubbard model is performed and the BAE for some subsectors of the theory are determined. In section 6, a perturbative treatment *à la* Klein and Seitz [20] of the obtained Hubbard-like Hamiltonians is performed; second order and fourth order terms are presented. The last section is devoted to a short conclusion. Finally, we give in Appendix A some hints for progressing towards integrable bosonic Hubbard models, and expose in Appendix B a twisted version of XX and Hubbard models, leading to Hamiltonians that depend on phases that can be identified with a Aharonov-Bohm phase.

2 Universal XX models

We generalize the construction given in [16, 19, 21] to the case of an arbitrary vector space \mathcal{V} , possibly infinite dimensional. We will use the standard auxiliary space notation, i.e. to any operator $A \in End(\mathcal{V})$, we associate the operator $A_1 = A \otimes \mathbb{I}$ and $A_2 = \mathbb{I} \otimes A$ in $End(\mathcal{V}) \otimes End(\mathcal{V})$. More generally, considering equalities in $End(\mathcal{V})^{\otimes k}$, A_j , $j = 1, \dots, k$, will act trivially in all spaces $End(\mathcal{V})$, but the j^{th} one.

To deal with superalgebras, we will also need a \mathbb{Z}_2 grading $[\cdot]$ on \mathcal{V} , such that $[v] = 0$ will be associated to bosonic states, $v \in \mathcal{V}_0$, and $[v] = 1$ to fermionic states, $v \in \mathcal{V}_1$.

We will also assume the existence of a (super-)trace operator, defined on a subset of $End(\mathcal{V})$ and obeying cyclicity. When \mathcal{V} is finite dimensional, $dim \mathcal{V} = \mathfrak{n}$, $End(\mathcal{V})$ is just the algebra $gl(\mathfrak{n})$, so that the trace operator is the usual trace of $\mathfrak{n} \times \mathfrak{n}$ matrices. If \mathcal{V} is graded and finite dimensional, one deals with the supertrace. When \mathcal{V} is infinite dimensional, the definition of a trace operator is more delicate, and one needs to verify that it exists and is cyclic for the operators we use. We address this problem in appendix A.

2.1 R-matrix

To define the R-matrix of universal XX model, we need some preliminary notions. We define projectors

$$\pi : \mathcal{V} \rightarrow \mathcal{W} \quad , \quad \bar{\pi} = \mathbb{I} - \pi : \mathcal{V} \rightarrow \overline{\mathcal{W}} \quad \text{with} \quad \mathcal{V} = \mathcal{W} \oplus \overline{\mathcal{W}} \quad (2.1)$$

and graded permutation operator

$$P_{12} : \begin{cases} \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V} \\ u \otimes v \rightarrow (-1)^{[u][v]} v \otimes u \end{cases} \quad (2.2)$$

Note that in auxiliary space notation, the action of the (graded) permutation operator reads

$$P_{12} u_1 v_2 = u_2 v_1 . \quad (2.3)$$

From these operators, one can construct an R-matrix

$$R_{12}(\lambda) = \Sigma_{12} P_{12} + \Sigma_{12} \sin \lambda + (\mathbb{I} \otimes \mathbb{I} - \Sigma_{12}) P_{12} \cos \lambda \quad (2.4)$$

where Σ_{12} is built on the projection operators:

$$\Sigma_{12} = \pi_1 \bar{\pi}_2 + \bar{\pi}_1 \pi_2 \quad (2.5)$$

It is easy to show that Σ_{12} is also a projector in $\mathcal{V} \otimes \mathcal{V}$: $(\Sigma_{12})^2 = \Sigma_{12}$.

Let us introduce the parity operator C :

$$C = \pi - \bar{\pi} . \quad (2.6)$$

It obeys $C^2 = \mathbb{I}$ and is related to the R-matrix through the equalities

$$\Sigma_{12} = \frac{1}{2}(1 - C_1 C_2) \quad \text{and} \quad \mathbb{I} \otimes \mathbb{I} - \Sigma_{12} = \frac{1}{2}(1 + C_1 C_2) \quad (2.7)$$

that allow us to rewrite the R-matrix as

$$R(\lambda) = \cos\left(\frac{\lambda}{2}\right) \left(\cos\left(\frac{\lambda}{2}\right) P_{12} + \sin\left(\frac{\lambda}{2}\right) \mathbb{I} \otimes \mathbb{I} \right) - \sin\left(\frac{\lambda}{2}\right) C_1 C_2 \left(\sin\left(\frac{\lambda}{2}\right) P_{12} + \cos\left(\frac{\lambda}{2}\right) \mathbb{I} \otimes \mathbb{I} \right) . \quad (2.8)$$

One has

Theorem 2.1 *For all spaces \mathcal{V} and projectors π , the R-matrix (2.4) satisfies the following properties:*

– *Parity invariance:*

$$C_1 C_2 R_{12}(\lambda) = R_{12}(\lambda) C_1 C_2 \quad (2.9)$$

– *Sign transformation:*

$$R_{12}(-\lambda) = C_1 R_{12}(\lambda) C_2 \quad (2.10)$$

– *Symmetry*:

$$R_{12}(\lambda) = R_{21}(\lambda) \quad (2.11)$$

– *Unitarity*:

$$R_{12}(\lambda) R_{21}(-\lambda) = (\cos^2 \lambda) \mathbb{I} \otimes \mathbb{I} \quad (2.12)$$

– *Regularity* :

$$R_{12}(0) = P_{12} \quad (2.13)$$

– *Exchange relation*:

$$R_{12}(\lambda) R_{21}(\mu) = R_{12}(\mu) R_{21}(\lambda) \quad (2.14)$$

– *Yang–Baxter equation (YBE)*:

$$\begin{aligned} R_{12}(\lambda_{12}) R_{13}(\lambda_{13}) R_{23}(\lambda_{23}) &= R_{23}(\lambda_{23}) R_{13}(\lambda_{13}) R_{12}(\lambda_{12}) \\ \text{where } \lambda_{ij} &= \lambda_i - \lambda_j. \end{aligned} \quad (2.15)$$

– *Decorated Yang–Baxter equation (dYBE)*:

$$\begin{aligned} R_{12}(\lambda'_{12}) C_1 R_{13}(\lambda_{13}) R_{23}(\lambda'_{23}) &= R_{23}(\lambda'_{23}) R_{13}(\lambda_{13}) C_1 R_{12}(\lambda'_{12}) \\ \text{with } \lambda'_{ij} &= \lambda_i + \lambda_j. \end{aligned} \quad (2.16)$$

Proof: The proof is strictly similar to the one done in [16], the only needed relations being

$$C^2 = \mathbb{I} \quad ; \quad C_1 \Sigma_{12} = \Sigma_{12} C_1 = -\Sigma_{12} C_2 = -C_2 \Sigma_{12} \quad (2.17)$$

and the relation (2.7). Let us also remark that this latter relation is equivalent to the relations

$$2 \Sigma_{ij} \Sigma_{kj} = \Sigma_{ij} + \Sigma_{kj} - \Sigma_{ik}, \quad \forall i, j, k \quad \text{with} \quad \Sigma_{ii} = 0 \quad (2.18)$$

without any reference to the projectors π and $\bar{\pi}$. However, a detailed analysis of the relations (2.18) shows that Σ_{12} must be of the form (2.7), up to conjugation. ■

Remark 2.1 When $\pi = 0$ or $\pi = \mathbb{I}$, we get $R_{12} = \cos(\lambda) P_{12}$, which also obeys all the statements of theorem 2.1, but leads to trivial models.

Lemma 2.2 *If we denote by $R_{12}^{(\pi)}(\lambda)$ the R -matrix built on π , we have*

$$R_{12}^{(\pi)}(\lambda) = R_{12}^{(\mathbb{I}-\pi)}(\lambda).$$

Proof: We have $C^{(\pi)} = -C^{(\mathbb{I}-\pi)}$, leading to the property $\Sigma_{12}^{(\pi)} = \Sigma_{12}^{(\mathbb{I}-\pi)}$. ■

2.2 Monodromy and transfer matrices

From the R-matrix, one constructs the (L sites) monodromy matrix

$$\mathcal{L}_{0<1\dots L>}(\lambda) = R_{01}(\lambda) R_{02}(\lambda) \cdots R_{0L}(\lambda) \quad (2.19)$$

which obeys the relation

$$R_{00'}(\lambda - \mu) \mathcal{L}_{0<1\dots L>}(\lambda) \mathcal{L}_{0'<1\dots L>}(\mu) = \mathcal{L}_{0'<1\dots L>}(\mu) \mathcal{L}_{0<1\dots L>}(\lambda) R_{00'}(\lambda - \mu). \quad (2.20)$$

This relation allows us to construct an (L sites) integrable XX spin chain through the transfer matrix

$$t_{1\dots L}(\lambda) = \text{tr}_0 \mathcal{L}_{0<1\dots L>}(\lambda) = \text{tr}_0 \left(R_{01}(\lambda) R_{02}(\lambda) \cdots R_{0L}(\lambda) \right). \quad (2.21)$$

Indeed, when the trace operator is well-defined on the monodromy matrix², the relation (2.20) implies that the transfer matrices for different values of the spectral parameter commute

$$[t_{1\dots L}(\lambda), t_{1\dots L}(\mu)] = 0. \quad (2.22)$$

Then, the XX-Hamiltonian is defined by

$$H = t_{1\dots L}(0)^{-1} \frac{dt_{1\dots L}}{d\lambda}(0). \quad (2.23)$$

Since the R-matrix is regular, H is local:

$$H = \sum_{j=1}^L H_{j,j+1} \quad \text{with} \quad H_{j,j+1} = P_{j,j+1} \Sigma_{j,j+1} \quad (2.24)$$

where we have used periodic boundary conditions, i.e. identified the site $L+1$ with the first site.

2.3 Symmetry of universal XX models

Proposition 2.3 *Let us consider a universal XX model based on a vector space \mathcal{V} , with projectors $\pi : \mathcal{V} \rightarrow \mathcal{W}$ and $\bar{\pi} : \mathcal{V} \rightarrow \bar{\mathcal{W}}$. For $\mathbb{M} \in \text{End}(\mathcal{W}) \oplus \text{End}(\bar{\mathcal{W}})$, one has*

$$(\mathbb{M}_1 + \mathbb{M}_2) R_{12}(\lambda) = R_{12}(\lambda) (\mathbb{M}_1 + \mathbb{M}_2). \quad (2.25)$$

As a consequence, the transfer matrix also has a symmetry (super)algebra $\text{End}(\mathcal{W}) \oplus \text{End}(\bar{\mathcal{W}})$, with generators given by

$$\mathbb{M}_{<1\dots L>} = \mathbb{M}_1 + \mathbb{M}_2 + \dots + \mathbb{M}_L, \quad (2.26)$$

where $\mathbb{M} \in \text{End}(\mathcal{W}) \oplus \text{End}(\bar{\mathcal{W}})$. The same is true for any Hamiltonian H built on the transfer matrix.

²For finite dimensional vector spaces, the trace operator is obviously always defined. For infinite dimensional spaces, one needs to be more careful: we will come back on this point in appendix A.

Proof: Starting from a general morphism $\mathbb{M} \in \text{End}(\mathcal{V})$, a direct calculation shows that when $\mathbb{M}(\mathcal{W}) \subset \mathcal{W}$ and $\mathbb{M}(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$, we have $\mathbb{M}\pi = \pi\mathbb{M}$ and $\mathbb{M}\overline{\pi} = \overline{\pi}\mathbb{M}$ so that (2.25) holds. The above conditions are equivalent to $\mathbb{M} \in \text{End}(\mathcal{W}) \oplus \text{End}(\overline{\mathcal{W}})$.

As far as the transfer matrix is concerned, the proof is the well-known, once (2.25) holds. ■

Since the choice of the projector π fixes \mathcal{W} and $\overline{\mathcal{W}}$, the above procedure allows us to associate to any symmetry (super)algebra \mathcal{S} a universal XX model possessing \mathcal{S} as symmetry.

The eigenstates of the transfer matrix will be also eigenstates of the Cartan generators of the symmetry algebra. These generators are given by \mathbb{M}_{aa} , $a = 1, \dots, \dim \mathcal{V} = d$ (with possibly $d = \infty$). The corresponding charges will be noted $\Lambda = (\lambda_1, \dots, \lambda_d)$. The charges $(\lambda_1, \dots, \lambda_r)$, $r = \text{rank} \pi$, correspond to $\text{End}(\mathcal{W})$, while $(\lambda_{r+1}, \dots, \lambda_d)$ are associated to $\text{End}(\overline{\mathcal{W}})$. In the following, we will also need the fundamental weights

$$\Lambda_a = (\underbrace{0, \dots, 0}_{a-1}, 1, 0, \dots, 0)^t, \quad a = 1, \dots, \dim \mathcal{V} = d. \quad (2.27)$$

3 Universal XX models based on $gl(\mathfrak{m}|\mathfrak{n})$

3.1 Hamiltonian and transfer matrix

Projectors and R -matrix: We apply the above construction to the case where \mathcal{V} is the graded tensor product $\mathcal{V} = \mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$, with possibly $\mathfrak{n} = 0$ to encompass the case $\mathcal{V} = \mathbb{C}^{\mathfrak{m}}$. In the following, we note $\mathfrak{s} = \mathfrak{n} + \mathfrak{m}$.

The \mathbb{Z}_2 grading $[.]$ is defined on indices j , such that $[j] = 0$, $1 \leq j \leq \mathfrak{m}$, will be associated to bosons and $[j] = 1$, $\mathfrak{m} + 1 \leq j \leq \mathfrak{m} + \mathfrak{n}$ to fermions. Accordingly, the elementary matrices E_{ij} (with 1 at position (i, j) and 0 elsewhere) will have grade $[E_{ij}] = [i] + [j]$.

To define the projectors π and $\overline{\pi}$, we introduce a subset

$$\mathcal{N} \subset \mathbb{Z}_{\mathfrak{s}} = [1, \mathfrak{s}] \cap \mathbb{Z}_+,$$

and denote by $\overline{\mathcal{N}}$ its complementary set, i.e.

$$\mathcal{N} \cap \overline{\mathcal{N}} = \emptyset \quad \text{and} \quad \mathcal{N} \cup \overline{\mathcal{N}} = \mathbb{Z}_{\mathfrak{s}}.$$

We will also need the bosonic and fermionic ‘components’ of \mathcal{N} ,

$$\mathcal{N}_0 = \mathcal{N} \cap \mathbb{Z}_{\mathfrak{m}} \quad \mathcal{N}_1 = \mathcal{N} \setminus \mathcal{N}_0 \quad \text{with} \quad \mathcal{N}_0 \cup \mathcal{N}_1 = \mathcal{N}.$$

They are such that $[j] = 0$ when $j \in \mathcal{N}_0$ while $[j] = 1$ when $j \in \mathcal{N}_1$.

To each set \mathcal{N} , one associates projectors

$$\pi^{(\mathcal{N})} = \sum_{j \in \mathcal{N}} E_{jj} \quad , \quad \overline{\pi} = \mathbb{I}_{\mathfrak{s}} - \pi = \pi^{(\overline{\mathcal{N}})} \quad (3.1)$$

Although these projectors depend on the set \mathcal{N} , we will drop the superscript (\mathcal{N}) , keeping it only when several sets \mathcal{N} are considered.

From these projectors, one constructs the R-matrix according to the general formulas (2.4) and (2.5). This R -matrix obeys theorem 2.1, with the parity matrix C :

$$C = \sum_{j \in \mathcal{N}} E_{jj} - \sum_{k \in \overline{\mathcal{N}}} E_{kk} = \pi - \overline{\pi}. \quad (3.2)$$

Monodromy matrix and Hamiltonian: From the R-matrix, one constructs the (L sites) monodromy and transfer matrices following the general procedure explained in section 2.2.

Then, the XX-Hamiltonian is defined by eq. (2.24) with two sites Hamiltonian

$$H_{j,j+1} = \sum_{i \in \mathcal{N}} \sum_{\bar{a} \in \overline{\mathcal{N}}} \left((-1)^{[\bar{a}]} E_{i\bar{a}} \otimes E_{\bar{a}i} + (-1)^{[i]} E_{\bar{a}i} \otimes E_{i\bar{a}} \right). \quad (3.3)$$

Anticipating the Bethe ansatz analysis, one can see that this Hamiltonian describes, apart from the ‘vacuum’, $\mathfrak{m} + \mathfrak{n} - 1$ species of particles gathered into two subsets, so-called the ‘barred’ \bar{a}, \bar{b}, \dots and ‘unbarred’ particles a, b, \dots corresponding to projectors $\overline{\pi}$ and $\pi = \mathbb{I}_{\mathfrak{s}} - \overline{\pi}$ respectively. The ‘barred’ particles move as hard-core particles while the ‘unbarred’ particles are displaced by the barred ones. This latter property is valid for a vacuum of ‘unbarred’ type: obviously, one has to reverse ‘barred’ and ‘unbarred’ particles if the vacuum is chosen of ‘barred’ type.

Symmetry and number of models: Obviously, without any loss of generality, one can choose

$$\mathcal{N} = \{1, 2, \dots, \mathfrak{r}_0; \mathfrak{m} + 1, \mathfrak{m} + 2, \dots, \mathfrak{m} + \mathfrak{r}_1\} \quad \text{with} \quad \mathfrak{r}_0 = |\mathcal{N}_0|, \quad \mathfrak{r}_1 = |\mathcal{N}_1| \quad (3.4)$$

$$\overline{\mathcal{N}} = \{\mathfrak{r}_0 + 1, \mathfrak{r}_0 + 2, \dots, \mathfrak{m}; \mathfrak{m} + \mathfrak{r}_1 + 1, \mathfrak{m} + \mathfrak{r}_1 + 2, \dots, \mathfrak{m} + \mathfrak{n} = \mathfrak{s}\} \quad (3.5)$$

From the property $R_{12}^{(\mathcal{N})}(\lambda) = R_{12}^{(\overline{\mathcal{N}})}(\lambda)$ and the isomorphism $gl(\mathfrak{n}|\mathfrak{m}) \simeq gl(\mathfrak{m}|\mathfrak{n})$, one can impose the inequalities

$$\mathfrak{r}_0 = |\mathcal{N}_1| \leq \frac{\mathfrak{m} + 1}{2} \quad \text{and} \quad \mathfrak{m} \geq \mathfrak{n},$$

leading to $\left(\left\lceil \frac{\max + 1}{2} \right\rceil + 1\right) (\min + 1)$ different models, where we used the notation $\min = \min(\mathfrak{n}, \mathfrak{m})$ and $\max = \max(\mathfrak{n}, \mathfrak{m})$.

The R-matrix admits a $gl(\mathfrak{m} - \mathfrak{r}_0|\mathfrak{n} - \mathfrak{r}_1) \oplus gl(\mathfrak{r}_0|\mathfrak{r}_1)$ symmetry superalgebra whose generators have the form

$$\begin{aligned} E_{jk}, \quad j, k \in \mathcal{N} & \quad \text{for} \quad gl(\mathfrak{r}_0|\mathfrak{r}_1) \\ E_{jk}, \quad j, k \in \overline{\mathcal{N}} & \quad \text{for} \quad gl(\mathfrak{m} - \mathfrak{r}_0|\mathfrak{n} - \mathfrak{r}_1). \end{aligned} \quad (3.6)$$

As a consequence, the transfer matrix also admits $gl(\mathfrak{m} - \mathfrak{r}_0|\mathfrak{n} - \mathfrak{r}_1) \oplus gl(\mathfrak{r}_0|\mathfrak{r}_1)$ symmetry superalgebra, with generators given by

$$\mathbb{M}_{<1 \dots L>} = \mathbb{M}_1 + \mathbb{M}_2 + \dots + \mathbb{M}_L, \quad (3.7)$$

where \mathbb{M} is one of the generators given in (3.6). The same is true for any Hamiltonian H built on the transfer matrix.

Since the choice of the projector π fixes the values of \mathfrak{r}_0 and \mathfrak{r}_1 , the above procedure allows us to associate to any symmetry (super)algebra $\mathcal{S} = gl(\mathfrak{q}|\mathfrak{q}') \oplus gl(\mathfrak{m} - \mathfrak{q}|\mathfrak{n} - \mathfrak{q}')$ a generalized XX model possessing \mathcal{S} as symmetry, provided the vector space $\mathcal{V} = \mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ we start from is large enough (i.e. $\mathfrak{m} \geq \mathfrak{q}$ and $\mathfrak{n} \geq \mathfrak{q}'$ to get \mathcal{S}). Conversely, from the vector space $\mathcal{V} = \mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$, one can construct models possessing the symmetry:

$$gl(\mathfrak{q}|\mathfrak{q}') \oplus gl(\mathfrak{m} - \mathfrak{q}|\mathfrak{n} - \mathfrak{q}'), \quad \mathfrak{q} \leq \mathfrak{m} \quad \text{and} \quad \mathfrak{q}' \leq \mathfrak{n} \quad (3.8)$$

3.2 BAEs for universal XX models

To get the BAEs of a model, one starts with a reference state, called the pseudo-vacuum, which is an eigenvector of the transfer matrix. The other states are constructed as ‘pseudo-excitations’ on this pseudo-vacuum.

3.2.1 The pseudo-vacua sector

The full space of states for the XX models is $(\mathcal{V})^{\otimes L}$: we consider here the subspace $\mathcal{W}_{vac} = (\mathcal{W})^{\otimes L}$. In this subspace, the transfer matrix takes a simple form:

$$t_{XX}(\lambda) \Big|_{\mathcal{W}_{vac}} = (\cos \lambda)^L P_{1L} P_{2L} \dots P_{L-1,L} + (\sin \lambda)^L \bar{\mathfrak{r}} \quad \text{with} \quad \bar{\mathfrak{r}} = \text{rank} \pi \quad (3.9)$$

One recognizes in $t_{XX}(0) = \exp(\widehat{i\mathfrak{p}})$ the shift automorphism. The eigenvalues of $\widehat{\mathfrak{p}}$ are the impulsions of the states. Note that the Hamiltonian $H_{XX} = \ln(t)'(0)$ (given in (2.24)) vanishes on this subspace.

There are a priori \mathfrak{r} reference states

$$\Omega_a = (e_a)^{\otimes L}, \quad a = 1, \dots, \mathfrak{r} = \text{rank} \pi \quad (3.10)$$

which have vanishing impulsion and charge $L \Lambda_a$ where Λ_a is the fundamental weight given in (2.27). However, since the algebra $\mathcal{S} = \text{End}(\mathcal{W}) \oplus \text{End}(\overline{\mathcal{W}})$ is a symmetry of the model, one can restrict itself to highest weight vectors and get the remaining states through the action of the step generators of \mathcal{S} . In fact, in (3.10), there is a unique highest weight vector

$$\Omega_1 = e_1 \otimes \dots \otimes e_1. \quad (3.11)$$

The other states in (3.10) can be obtained through iterative action of the symmetry generators $\mathbb{M}_{a,1}$:

$$\Omega_a = (\mathbb{M}_{a,1})^L \Omega_1 \quad (3.12)$$

In the following, we will take Ω_1 is as the vacuum. The other states will be described as excitations above this vacuum, and we introduce for M indices b_1, \dots, b_M , and M positions x_1, \dots, x_M the state:

$$|\{b\}; \mathbf{x}\rangle = \underbrace{e_1 \otimes \dots \otimes e_1}_{x_1-1} \otimes e_{b_1} \otimes \underbrace{e_1 \otimes \dots \otimes e_1}_{x_2-x_1-1} \otimes e_{b_2} \otimes e_1 \otimes \dots \otimes e_1 \otimes e_{b_M} \otimes e_1 \otimes \dots \otimes e_1 \quad (3.13)$$

3.2.2 One excitation states

We introduce

$$\Phi_a^1(p) = \sum_{x=1}^L e^{ipx} |a, x\rangle \quad a = 2, \dots, \mathfrak{r} \text{ rank}(\pi) = \mathfrak{r} \quad (3.14)$$

$$\Phi_{\bar{a}}^1(p) = \sum_{x=1}^L e^{ipx} |\bar{a}, x\rangle \quad \bar{a} = \mathfrak{r} + 1, \dots, \mathfrak{s} = \mathfrak{r} + \bar{\mathfrak{r}} = \mathfrak{n} + \mathfrak{m} \quad (3.15)$$

where $|a, x\rangle$ is defined as in (3.13). The indices $a = 2, \dots, \mathfrak{r}$ correspond to the space \mathcal{W} and the indices $\bar{a} = \mathfrak{r} + 1, \dots, \mathfrak{s} = \mathfrak{r} + \bar{\mathfrak{r}}$ correspond to the space $\overline{\mathcal{W}}$. Through a direct calculation, it is easy to show that

$$t(0) \Phi_\alpha^1(p) = e^{ip} \Phi_\alpha^1(p), \quad \alpha = a, \bar{a} \quad (3.16)$$

$$H \Phi_a^1(p) = 0 \quad \text{and} \quad H \Phi_{\bar{a}}^1(p) = 2 \cos(p) \Phi_{\bar{a}}^1(p) \quad (3.17)$$

if p obeys the Bethe ansatz equation (BAE)

$$e^{ipL} = 1 \quad (3.18)$$

One can gather all these states into a single vector state. The set $\{a = 2, \dots, \mathfrak{r}; \bar{a} = \mathfrak{r} + 1, \dots, \mathfrak{s} = \mathfrak{r} + \bar{\mathfrak{r}}\}$ is noted $\{j = 1, \dots, \mathfrak{s} - 1\}$ where the first $\mathfrak{r} - 1$ indices are of type ‘ a ’ while the $\bar{\mathfrak{r}}$ last ones are of type ‘ \bar{a} ’. We introduce the elementary vectors $u_j \in \mathbb{C}^{\mathfrak{s}-1}$ (with 1 at position j and 0 elsewhere): they correspond to the ‘small’ chain of the nested Bethe ansatz. The vector state reads:

$$\Phi^1(p) = \sum_{j=1}^{\mathfrak{s}-1} \Phi_{j+1}^1(p) u_j = \sum_{x=1}^L e^{ipx} |x\rangle \quad \text{with} \quad |x\rangle = \sum_{j=1}^{\mathfrak{s}-1} |j, x\rangle u_j \quad (3.19)$$

Note that in $|x\rangle$, $|j, x\rangle$ lies on the original ‘big’ chain (of length L), while u_j lies on a new ‘small’ chain (here of length 1). The basic idea is to ‘move’ the action of the transfer matrix and symmetry generators from the ‘big chain’ to the ‘small one’. Indeed, we have

$$t(0) \Phi^1(p) = e^{ip} \Phi^1(p) \quad (3.20)$$

$$H \Phi^1(p) = D(p) \Phi^1(p) \quad \text{with} \quad D(p) = 2 \cos(p) \text{diag}(\underbrace{0, \dots, 0}_{\mathfrak{r}-1}, \underbrace{1, \dots, 1}_{\bar{\mathfrak{r}}}) \quad (3.21)$$

The matrix $D(p)$ acts on the small chain (i.e. on the vectors u_j) while H was acting on the big chain (i.e. on the states $|j, x\rangle$). In the same way, the charges of the states are given by

$$\mathbb{M}_{j+1, j+1} \Phi^1(p) = E_{jj} \Phi^1(p), \quad j = 1, \dots, \mathfrak{s} - 1 \quad (3.22)$$

$$\mathbb{M}_{11} \Phi^1(p) = (L - 1) \Phi^1(p) \quad (3.23)$$

where $E_{ij} \in \text{End}(\mathbb{C}^{\mathfrak{s}-1})$, $i, j > 1$, (the elementary matrix with 1 at position (i, j) and 0 elsewhere) acts on the small chain. It corresponds to the generator of the symmetry generator \mathbb{M}_{ij} acting on the big chain. \mathbb{M}_{11} (more precisely $L - \mathbb{M}_{11}$) acts as a scalar and corresponds to the excitation number.

3.2.3 Two excitation states and scattering matrix

We look for eigenstates $\Phi_{i,j}^2(p_1, p_2)$ describing two excitations of type i and j , and with impulsion p_1 and p_2 . We gather these states into a single vector

$$\Phi^2(p_1, p_2) = \sum_{i,j=1}^{\mathfrak{s}-1} \Phi_{i+1,j+1}^2(p_1, p_2) u_i \otimes u_j \quad (3.24)$$

defining a length 2 ‘small chain’ (carried by the vectors u_j). The construction is done in the following way:

$$\Phi^2(p_1, p_2) = \sum_{1 \leq x_1 < x_2 \leq L} \left\{ e^{i p \cdot x} \mathbb{I}_{\mathfrak{s}-1} \otimes \mathbb{I}_{\mathfrak{s}-1} + e^{i \gamma(p) \cdot x} P_{12} \mathcal{S}_{12}(p_1, p_2) \right\} |x_1, x_2\rangle \quad (3.25)$$

$$|x_1, x_2\rangle = \sum_{i,j=1}^{\mathfrak{s}-1} |i+1, j+1; x_1, x_2\rangle u_i \otimes u_j \quad (3.26)$$

$$\mathcal{S}_{12}(p_1, p_2) = e^{-ip_1} \overset{\circ}{\pi} \otimes \bar{\pi} + e^{ip_2} \bar{\pi} \otimes \overset{\circ}{\pi} - P_{12} \left(\overset{\circ}{\pi} \otimes \overset{\circ}{\pi} + \bar{\pi} \otimes \bar{\pi} \right) \quad (3.27)$$

where we have introduced $p \cdot x = p_1 x_1 + p_2 x_2$ and $\gamma(p) \cdot x = p_2 x_1 + p_1 x_2$. P_{12} is the (graded) permutation, $\bar{\pi}$ is the projector on $\overline{\mathcal{W}}$ and $\overset{\circ}{\pi} = \mathbb{I}_{\mathfrak{s}-1} - \bar{\pi}$. Since the scattering matrix \mathcal{S} acts on the small chain, the projector $\overset{\circ}{\pi}$ is the (lower rank) counter part in the small chain of the projector π (that acts in the big chain). We kept the same notation $\bar{\pi}$ for both of the projectors $\mathbb{I}_{\mathfrak{s}} - \pi$ and $\mathbb{I}_{\mathfrak{s}-1} - \overset{\circ}{\pi}$ because they are obviously isomorphic. This asymmetric situation is due to our choice of the vacuum, that belongs to $\mathcal{W} = \pi(\mathcal{V})$, not to $\overline{\mathcal{W}}$.

The scattering matrix obeys Yang-Baxter equation and unitarity relation

$$\mathcal{S}_{12}(p_1, p_2) \mathcal{S}_{13}(p_1, p_3) \mathcal{S}_{23}(p_2, p_3) = \mathcal{S}_{23}(p_2, p_3) \mathcal{S}_{13}(p_1, p_3) \mathcal{S}_{12}(p_1, p_2), \quad (3.28)$$

$$\mathcal{S}_{12}(p_1, p_2) \mathcal{S}_{21}(p_2, p_1) = \mathbb{I}_{\mathfrak{s}-1} \otimes \mathbb{I}_{\mathfrak{s}-1}, \quad (3.29)$$

$$\mathcal{S}_{21}(p_1, p_2) = P_{12} \mathcal{S}_{12}(p_1, p_2) P_{12}, \quad (3.30)$$

while the braided S -matrix $\check{\mathcal{S}}_{12}(p_1, p_2) = P_{12} \mathcal{S}_{12}(p_1, p_2)$ (which appears in $\Phi^2(p_1, p_2)$) obeys braided Yang-Baxter equation and braided unitarity relation:

$$\check{\mathcal{S}}_{23}(p_1, p_2) \check{\mathcal{S}}_{12}(p_1, p_3) \check{\mathcal{S}}_{23}(p_2, p_3) = \check{\mathcal{S}}_{12}(p_2, p_3) \check{\mathcal{S}}_{23}(p_1, p_3) \check{\mathcal{S}}_{12}(p_1, p_2), \quad (3.31)$$

$$\check{\mathcal{S}}_{12}(p_1, p_2) \check{\mathcal{S}}_{12}(p_2, p_1) = \mathbb{I}_{\mathfrak{s}-1} \otimes \mathbb{I}_{\mathfrak{s}-1}. \quad (3.32)$$

It is easy to show that

$$t(0) \Phi^2(p_1, p_2) = e^{i(p_1+p_2)} \Phi^2(p_1, p_2), \quad (3.33)$$

$$H \Phi^2(p_1, p_2) = D(p_1, p_2) \Phi^2(p_1, p_2) \quad \text{with} \quad D(p_1, p_2) = D(p_1) \otimes \mathbb{I}_{\mathfrak{s}-1} + \mathbb{I}_{\mathfrak{s}-1} \otimes D(p_2)$$

if the BAEs

$$e^{ip_2 L} \Phi^2(p_1, p_2) = \mathcal{S}_{12}(p_1, p_2) \Phi^2(p_1, p_2) \quad (3.34)$$

$$e^{ip_1 L} \Phi^2(p_1, p_2) = \mathcal{S}_{21}(p_2, p_1) \Phi^2(p_1, p_2) \quad (3.35)$$

are satisfied. We have used $D(p)$ defined in (3.21), leading to energies $0, 2 \cos(p_1), 2 \cos(p_2)$ and $2 \cos(p_1) + 2 \cos(p_2)$. The charges of the states are given by

$$\mathbb{M}_{j+1,j+1} \Phi^2(p_1, p_2) = \left(E_{jj} \otimes \mathbb{I}_{\mathfrak{s}-1} + \mathbb{I}_{\mathfrak{s}-1} \otimes E_{jj} \right) \Phi^2(p_1, p_2), \quad j = 1, \dots, \mathfrak{s} - 1 \quad (3.36)$$

$$\mathbb{M}_{11} \Phi^2(p_1, p_2) = (L - 2) \Phi^2(p_1, p_2) \quad (3.37)$$

Again, the action of the Hamiltonian H and symmetry generators \mathbb{M}_{jj} , $j > 1$ have been ‘moved’ to matrices acting on the small chain. $L - \mathbb{M}_{11}$ is the excitation number. Since all the matrices are diagonal, we have indeed eigenvectors of the Hamiltonian and symmetry generators.

Remark the property

$$\Phi^2(p_2, p_1) = \check{S}_{12}(p_1, p_2)^{-1} \Phi^2(p_1, p_2) = \check{S}_{12}(p_2, p_1) \Phi^2(p_1, p_2) \quad (3.38)$$

that ensures that we can impose $p_1 < p_2$.

The explicit form of the BAE depends on the type of excitation one considers. Looking at their projection on vectors $u_i \otimes u_j$ with $i, j < \mathfrak{r}$, one gets the BAE for type a, b excitations:

$$e^{ip_j L} = \omega, \quad j = 1, 2 \quad \text{with} \quad \omega^2 = 1 \quad (3.39)$$

If one projects on $u_i \otimes u_j$ with $i, j \geq \mathfrak{r}$, one gets the BAE for type \bar{a}, \bar{b} excitations:

$$e^{ip_j L} = \omega, \quad j = 1, 2 \quad (3.40)$$

If one projects on $u_i \otimes u_j$ with $i < \mathfrak{r}$ and $j \geq \mathfrak{r}$, one gets the BAE for type a, \bar{a} excitations:

$$e^{ip_1(L-1)} = 1 \quad \text{and} \quad e^{i(p_1+p_2)L} = 1 \quad (3.41)$$

where p_1 is attached to the type a excitation.

If one projects on $u_i \otimes u_j$ with $i \geq \mathfrak{r}$ and $j < \mathfrak{r}$, one gets the BAE for type \bar{a}, a excitations:

$$e^{ip_2(L-1)} = 1 \quad \text{and} \quad e^{i(p_1+p_2)L} = 1 \quad (3.42)$$

where p_2 is attached to the type a excitation.

3.2.4 M excitation states and BAEs

We consider general states $\Phi_{\{j\}}^M(\mathbf{p})$, with M excitations of momenta p_m , $m = 1, \dots, M$, gathered into a vector \mathbf{p} . If M' is the total number of type ‘unbarred’ excitations, we note their corresponding momentum q_n , $n = 1, \dots, M'$, gathered in a vector \mathbf{q} . In the same way, for $M'' = M - M'$ the total number of type ‘barred’ excitations, we noted \bar{q}_n , $n = 1, \dots, M''$ their momentum, gathered in $\bar{\mathbf{q}}$. Hence we have

$$\{p_1, p_2, \dots, p_M\} = \{q_1, q_2, \dots, q_{M'}\} \cup \{\bar{q}_1, \bar{q}_2, \dots, \bar{q}_{M''}\}. \quad (3.43)$$

Then, the state $\Phi_{\{j\}}^M(\mathbf{p})$ is characterized by:

$$\Phi_{\{j\}}^M(\mathbf{p}) : \begin{cases} M \text{ excitations above the vacuum} \\ \text{Momentum: } |\mathbf{p}| = \sum_{m=1}^M p_m \equiv |\mathbf{q}| + |\bar{\mathbf{q}}| = \sum_{n=1}^{M'} q_n + \sum_{n=1}^{M''} \bar{q}_n \\ \text{Charge w.r.t. the symmetry algebra: } Q = \sum_{m=1}^M \Lambda_{j_m} \\ \text{Energy: } E = \sum_{n=1}^{M''} 2 \cos(\bar{q}_n) \end{cases} \quad (3.44)$$

where the weights Λ_j are given in (2.27).

All the states with M excitations can be gathered into a single vector

$$\Phi^M(\mathbf{p}) = \sum_{\{j\}} \Phi_{\{j\}}^M(\mathbf{p}) u_{j_1} \otimes u_{j_2} \otimes \dots \otimes u_{j_M} \in \underbrace{\mathbb{C}^{m-1|n} \otimes \dots \otimes \mathbb{C}^{m-1|n}}_M \quad (3.45)$$

describing a small chain of length M . Then, as for one and two excitation states, the action of the different integrals of motion can be ‘transferred’ from the original (‘big’) chain to the new ‘small’ chain:

$$t(0) \Phi^M(\mathbf{p}) = e^{i|\mathbf{p}|} \Phi^M(\mathbf{p}) \quad (3.46)$$

$$H \Phi^M(\mathbf{p}) = D(\mathbf{p}) \Phi^M(\mathbf{p}) \quad (3.47)$$

$$D(\mathbf{p}) = \sum_{m=1}^M \underbrace{\mathbb{I}_{s-1} \otimes \dots \otimes \mathbb{I}_{s-1}}_{m-1} \otimes D(p_m) \otimes \underbrace{\mathbb{I}_{s-1} \otimes \dots \otimes \mathbb{I}_{s-1}}_{M-m} \equiv \sum_{m=1}^M D_m(p_m) \quad (3.48)$$

$$\mathbb{M}_{jj} \Phi^M(\mathbf{p}) = \mathbb{E}_{j-1,j-1} \Phi^M(\mathbf{p}), \quad j = 2, \dots, \mathfrak{r} + \bar{\mathfrak{r}} \quad (3.49)$$

$$\mathbb{E}_{jj} = \sum_{m=1}^M \underbrace{\mathbb{I}_{s-1} \otimes \dots \otimes \mathbb{I}_{s-1}}_{m-1} \otimes E_{jj} \otimes \underbrace{\mathbb{I}_{s-1} \otimes \dots \otimes \mathbb{I}_{s-1}}_{M-m} \equiv \sum_{m=1}^M E_{jj}^{(m)} \quad (3.50)$$

$$\mathbb{M}_{11} \Phi^M(\mathbf{p}) = (L - M) \Phi^M(\mathbf{p}). \quad (3.51)$$

In equalities (3.46)-(3.51), the left-hand sides correspond to action on the original chain, while the right-hand sides corresponds to their counter-part on the ‘small’ chain. All the matrices in the r.h.s. are diagonal, and the projection of these r.h.s. on a generic state $u_{j_1} \otimes u_{j_2} \otimes \dots \otimes u_{j_M}$ reproduces the data (3.44).

The BAEs of the model take the form

$$e^{ip_j L} \Phi^M(\mathbf{p}) = \mathcal{S}_{j+1,j} \mathcal{S}_{j+2,j} \dots \mathcal{S}_{Mj} \mathcal{S}_{1j} \mathcal{S}_{2j} \dots \mathcal{S}_{j-1,j} \Phi^M(\mathbf{p}) \quad j = 1, \dots, M \quad (3.52)$$

where $\mathcal{S}_{jk} \equiv \mathcal{S}_{jk}(p_j, p_k)$ is the two-body scattering matrix (3.27) acting in the spaces j and k of the tensor product explicited in (3.45). To compute them explicitly, we introduce the order \prec defined by

$$j+1 \prec j+2 \prec \dots \prec M \prec 1 \prec 2 \prec \dots \prec j-1 \quad (3.53)$$

Any set of indices $\{j_1, j_2, \dots, j_n\}$ ordered accordingly, $j_1 \prec j_2 \prec \dots \prec j_n$, will be noted $[j_1, j_2, \dots, j_n]_{\prec}$. Then, from the form of \mathcal{S}_{12} , one computes

$$\begin{aligned} \mathcal{S}_{j+1,j} \mathcal{S}_{j+2,j} \dots \mathcal{S}_{Mj} \mathcal{S}_{1j} \mathcal{S}_{2j} \dots \mathcal{S}_{j-1,j} &= \sum_{n=0}^M \sum_{\{j\}_{\prec}^{\circ} \oplus \{k\}} (-1)^n P_{jj_1} P_{jj_2} \dots P_{jj_n} \left\{ \right. \\ &\quad \overset{\circ}{\pi}_j \overset{\circ}{\pi}_{j_1} \dots \overset{\circ}{\pi}_{j_n} \bar{\pi}_{k_1} \dots \bar{\pi}_{k_{M-1-n}} \exp \left(i (M-1-n) p_j \right) \\ &\quad \left. + \bar{\pi}_j \bar{\pi}_{j_1} \dots \bar{\pi}_{j_n} \overset{\circ}{\pi}_{k_1} \dots \overset{\circ}{\pi}_{k_{M-1-n}} \exp \left(-i \sum_{\ell=1}^{M-1-n} p_{k_\ell} \right) \right\} \end{aligned} \quad (3.54)$$

Above, the sum on $\{j\}_{\prec}^{\circ} \oplus \{k\}$ runs on partitions of $[1, M] \setminus \{j\}$, where the first set (of cardinality n) is ordered according to \prec , $\{j\}_{\prec}^{\circ} = [j_1, j_2, \dots, j_n]_{\prec}$, while the second set $\{k\} = \{k_1, k_2, \dots, k_{M-1-n}\}$ is its complementary set. $\overset{\circ}{\pi}_k$ is the projector operator $\overset{\circ}{\pi}$ in space k (of the ‘small chain’).

Applying (3.54) to $\Phi^M(\mathbf{p})$ leads to the BAEs for the XX model. To compute them, we remark that the operator $P_{jj_1} P_{jj_2} \dots P_{jj_n}$ corresponds to the cyclic permutation of the spaces j, j_1, \dots, j_n in the small chain. Its eigenvalues are $(\omega_n)^k$, $k = 1, 2, \dots, n$, where $\omega_n = e^{2i\pi/n}$. Moreover, this operator commutes with diagonal matrix within the brackets of eq. (3.54) and does not change the type of excitation. Thus, the BAEs take the form

$$\exp \left(i q_n (L - M'') \right) = (-1)^{M'-1} (\omega_{M'})^n, \quad n = 1, 2, \dots, M' \quad (3.55)$$

$$\exp \left(i L \bar{q}_n \right) = (\omega_{M''})^n \exp \left(-i |\mathbf{q}| \right), \quad n = 1, 2, \dots, M'' \quad (3.56)$$

$$\text{with } (\omega_{M'})^{M'} = 1, \quad (\omega_{M''})^{M''} = 1 \quad \text{and} \quad |\mathbf{q}| = \sum_{n=1}^{M'} q_n$$

Remark that multiplying together all the BAEs one gets

$$\exp \left(i L |\mathbf{p}| \right) = 1. \quad (3.57)$$

4 Universal Hubbard models

Starting with universal XX models, one can build universal Hubbard models, in the same way it has been done for usual and super Hubbard models [4, 16]. To simplify the presentation, we present the construction in the case of $gl(\mathbf{m}|\mathbf{n})$, but obviously the results are valid for any universal Hubbard model.

4.1 R-matrices

4.1.1 R-matrix for universal Hubbard models

We start with the R -matrices of two universal XX models, $R_{12}^{\uparrow}(\lambda)$ and $R_{12}^{\downarrow}(\lambda)$, leaving in two different sets of spaces that we label by \uparrow and \downarrow . Let us stress that the two XX models can

be based on two different (graded) vector spaces \mathcal{V}_\uparrow and \mathcal{V}_\downarrow , with two different projectors π_\uparrow and π_\downarrow , associated to the sets \mathcal{N}_\uparrow and \mathcal{N}_\downarrow .

The Hubbard model is constructed from the coupling of these two XX models. Its R -matrix reads:

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) = R_{12}^{\uparrow}(\lambda_{12}) R_{12}^{\downarrow}(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12}) R_{12}^{\uparrow}(\lambda'_{12}) C_1^{\uparrow} R_{12}^{\downarrow}(\lambda'_{12}) C_1^{\downarrow} \quad (4.1)$$

where again $\lambda_{12} = \lambda_1 - \lambda_2$ and $\lambda'_{12} = \lambda_1 + \lambda_2$. The definition of the parameter $h'_{12} = h(\lambda_1) + h(\lambda_2)$ is given below. It is easy to show that this R -matrix is symmetric

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) = R_{21}^{\downarrow\uparrow}(\lambda_1, \lambda_2), \quad (4.2)$$

regular

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_1) = P_{12}^{\uparrow\downarrow} = P_{12}^{\uparrow} P_{12}^{\downarrow} \quad (4.3)$$

and obeys the unitarity relation

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) R_{21}^{\downarrow\uparrow}(\lambda_2, \lambda_1) = \left(\cos^4(\lambda_{12}) - \left(\frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12}) \right)^2 \right) \mathbb{I}_{12}^{\uparrow} \otimes \mathbb{I}_{12}^{\downarrow} \\ \text{where } \mathbb{I}_{12} = \mathbb{I} \otimes \mathbb{I} \quad (4.4)$$

Property 4.1 *When the function $h(\lambda)$ is given by*

$$\sinh(2h) = U \sin(2\lambda) \quad (4.5)$$

for some (free) parameter U , the R -matrix (4.1) obeys YBE:

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) R_{13}^{\uparrow\downarrow}(\lambda_1, \lambda_3) R_{23}^{\uparrow\downarrow}(\lambda_2, \lambda_3) = R_{23}^{\uparrow\downarrow}(\lambda_2, \lambda_3) R_{13}^{\uparrow\downarrow}(\lambda_1, \lambda_3) R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2). \quad (4.6)$$

In that case, the coefficient in (4.4) can be rewritten as

$$\cos^2(\lambda_{12}) \left(\cos^2(\lambda_{12}) - \left(\frac{\tanh(h_{12})}{\cos(\lambda'_{12})} \right)^2 \right) \quad (4.7)$$

where $h_{12} = h(\lambda_1) - h(\lambda_2)$.

Proof: Again, as remarked in [16], the proof relies only on the properties (2.17), (2.7) and follows the steps of the original proof by Shiroishi [22], in the same way it has been done for algebras in [4]. Hence, the choice of the projector does not affect it. Moreover, it was already noticed in [4] that one can couple two XX models based on different $gl(\mathfrak{m})$ algebras: this obviously extends to general (graded) vector spaces \mathcal{V} . ■

4.1.2 Gauge version of universal Hubbard models

As for the usual Hubbard model, one can introduce a gauged version of the above R-matrix. It is defined by

$$\mathcal{R}_{12}(\lambda_1, \lambda_2) = e^{\frac{1}{2}h_1 C_1^\dagger C_1^\dagger} e^{\frac{1}{2}h_2 C_2^\dagger C_2^\dagger} R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) e^{-\frac{1}{2}h_1 C_1^\dagger C_1^\dagger} e^{-\frac{1}{2}h_2 C_2^\dagger C_2^\dagger}$$

where $h_j = h(\lambda_j)$, $j = 1, 2$ (4.8)

Being a gauged version of the previous R-matrix, $\mathcal{R}_{12}(\lambda_1, \lambda_2)$ also obeys YBE, is unitary and regular. This gauged version is used in usual Hubbard model to make contact between the above construction and the Hubbard R-matrix as it has been originally built by Shastry.

4.2 Monodromy matrices, transfer matrices and Hamiltonians

We remind that for given vector spaces \mathcal{V}_\downarrow and \mathcal{V}_\uparrow , the different possible projectors π_\downarrow and π_\uparrow give different R-matrices with, as we shall see, a different symmetry (super)algebra.

We consider the ‘reduced’ monodromy matrix

$$L_{a<b_1\dots b_L>}(\lambda) = \mathcal{R}_{ab_1}(\lambda, 0) \dots \mathcal{R}_{ab_L}(\lambda, 0) \quad (4.9)$$

and, when the trace is well-defined, its transfer matrix

$$t(\lambda) = \text{tr}_a L_{a<b_1\dots b_L>}(\lambda)$$

Then, one gets

$$[H, t(\lambda)] = 0, \quad \forall \lambda, \quad \text{for } H = H(0) = t(0)^{-1} t'(0) \quad (4.10)$$

This ‘reduced’ monodromy matrix is just the one used to define the Hubbard model; one can compute

$$\mathcal{R}_{12}(\lambda, 0) = \frac{1}{\cosh(h)} I_1^{\uparrow\downarrow}(h) R_{12}^\uparrow(\lambda) R_{12}^\downarrow(\lambda) I_1^{\uparrow\downarrow}(h) \quad (4.11)$$

where

$$I_1^{\uparrow\downarrow}(h) = \cosh\left(\frac{h}{2}\right) \mathbb{I} \otimes \mathbb{I} + \sinh\left(\frac{h}{2}\right) C_1^\uparrow C_1^\downarrow \quad (4.12)$$

The explicit form of the Hubbard Hamiltonian reads

$$H = \sum_{j=1}^L H_{j,j+1} \quad (4.13)$$

with

$$H_{j,j+1} = \Sigma_{j,j+1}^\uparrow P_{j,j+1}^\uparrow + \Sigma_{j,j+1}^\downarrow P_{j,j+1}^\downarrow + U C_j^\uparrow C_j^\downarrow \quad (4.14)$$

where we have used periodic boundary conditions. One can see that the kinetics is dictated by the XX models: barred particles moves ‘almost freely’ with the noticeable exception that \bar{a}^\uparrow and \bar{b}^\uparrow (or \bar{a}^\downarrow and \bar{b}^\downarrow) cannot cross. Unbarred particles of type up (resp. down) are displaced by barred particles of same type. There is interaction only between ‘up’ and ‘down’ particles, and the sign of the interaction depends on their ‘bar’ or ‘unbar’ type.

4.3 Symmetries

We generalize the results obtained for $su(\mathfrak{m})$ Hubbard models (see for instance [4, 11]) and $gl(\mathfrak{m}|\mathfrak{n})$ Hubbard models [16].

Proposition 4.2 *The transfer matrix of generalized Hubbard models admits as symmetry (super)algebra*

$$End(\mathcal{V}_0^\uparrow) \oplus End(\mathcal{V}_1^\uparrow) \oplus End(\mathcal{V}_0^\downarrow) \oplus End(\mathcal{V}_1^\downarrow),$$

each of the $End(\mathcal{V}_0^\varepsilon) \oplus End(\mathcal{V}_1^\varepsilon)$, $\varepsilon = \uparrow, \downarrow$ corresponding to the symmetry of one XX model.

As a consequence this symmetry is also valid for the Hubbard Hamiltonian.

Proof: To prove this symmetry, it is sufficient to remark that

$$\mathbb{M}C = C\mathbb{M} \quad (4.15)$$

where $\mathbb{M} = \mathbb{M}^\uparrow + \mathbb{M}^\downarrow$ and $\mathbb{M}^\varepsilon \in End(\mathcal{V}_0^\varepsilon) \oplus End(\mathcal{V}_1^\varepsilon)$, $\varepsilon = \uparrow, \downarrow$. Thus, one gets

$$[R_{12}(\lambda, 0), \mathbb{M}_1^\uparrow + \mathbb{M}_2^\uparrow] = 0 = [R_{12}(\lambda, 0), \mathbb{M}_1^\downarrow + \mathbb{M}_2^\downarrow] \quad (4.16)$$

where $R_{12}(\lambda, 0)$ is the R -matrix of the universal Hubbard model.

As far as Hamiltonians and transfer matrices are concerned, the generators of the symmetry have the form

$$\mathbb{M}^\uparrow = \sum_{j=1}^L \mathbb{M}_j^\uparrow \quad \text{and} \quad \mathbb{M}^\downarrow = \sum_{j=1}^L \mathbb{M}_j^\downarrow \quad (4.17)$$

■

The eigenstates of the transfer matrix will be also eigenstate of the Cartan generators $\mathbb{M}_{aa}^\varepsilon$, $a = 1, \dots, \dim \mathcal{V}^\varepsilon = d^\varepsilon$, $\varepsilon = \uparrow, \downarrow$. The corresponding charges will be noted $\Lambda^\varepsilon = (\lambda_1^\varepsilon, \dots, \lambda_d^\varepsilon)$.

5 BAE for universal Hubbard models

We follow the same steps as in section 3.2.

5.1 Scattering matrix

5.1.1 Pseudo-vacua sector

The full space of states is now $(\mathcal{V}^\uparrow \otimes \mathcal{V}^\downarrow)^{\otimes L}$, and we consider the subspace $\mathcal{W}_{vac} = (\mathcal{W}^\uparrow \otimes \mathcal{W}^\downarrow)^{\otimes L}$. In this subspace, the Hubbard transfer matrix takes a factorized form:

$$t(\lambda) \Big|_{\mathcal{W}_{vac}} = (1 + \tanh(h))^L t_{XX}^\uparrow(\lambda) \Big|_{\mathcal{W}_{vac}^\uparrow} t_{XX}^\downarrow(\lambda) \Big|_{\mathcal{W}_{vac}^\downarrow} \quad (5.1)$$

where $h(\lambda)$ has been given in (4.5). The eigenstates of this sector also take a factorized form

$$\Phi_{\{j\}, \{j'\}}^{M, M'}(\mathbf{p}, \mathbf{p}') = \Phi_{\{j\}}^{M, \uparrow}(\mathbf{p}) \Phi_{\{j'\}}^{M', \downarrow}(\mathbf{p}') \quad (5.2)$$

with eigenvalues

$$t(\lambda) \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') = \mathcal{E}(\mathbf{p}, \mathbf{p}'; \lambda) \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') \quad (5.3)$$

$$\mathcal{E}(\mathbf{p}, \mathbf{p}'; \lambda) = (1 + \tanh(h))^L \left((\cos \lambda)^L e^{i|\mathbf{p}|} + (\sin \lambda)^L \bar{\mathbf{r}}^\uparrow \right) \left((\cos \lambda)^L e^{i|\mathbf{p}'|} + (\sin \lambda)^L \bar{\mathbf{r}}^\downarrow \right). \quad (5.4)$$

Above, we have introduced $\bar{\mathbf{r}}^\uparrow = \text{rank}(\bar{\pi}^\uparrow)$ and $\bar{\mathbf{r}}^\downarrow = \text{rank}(\bar{\pi}^\downarrow)$. The charges of the states read

$$\mathbb{M}^\uparrow \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') = \Lambda^\uparrow \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') \quad \text{with} \quad \Lambda^\uparrow = (L - M) \Lambda_1^\uparrow + \sum_{m=1}^M \Lambda_{j_m}^\uparrow \quad (5.5)$$

$$\mathbb{M}^\downarrow \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') = \Lambda^\downarrow \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') \quad \text{with} \quad \Lambda^\downarrow = (L - M') \Lambda_1^\downarrow + \sum_{m=1}^{M'} \Lambda_{j'_m}^\downarrow. \quad (5.6)$$

Their momentum is given by

$$\begin{aligned} \hat{\mathbf{p}} \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') &= i \ln \mathcal{E}(\mathbf{p}, \mathbf{p}'; 0) \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') = \left(\sum_{m=1}^M p_m + \sum_{m=1}^{M'} p'_m \right) \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') \\ &\equiv (|\mathbf{p}| + |\mathbf{p}'|) \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') \end{aligned} \quad (5.7)$$

and their energy reads

$$H \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') = \frac{[\frac{d}{d\lambda} \mathcal{E}(\mathbf{p}, \mathbf{p}'; \lambda)]_{\lambda=0}}{\mathcal{E}(\mathbf{p}, \mathbf{p}'; 0)} \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') = U L \Phi_{\{j\},\{j'\}}^{MM'}(\mathbf{p}, \mathbf{p}') \quad (5.8)$$

5.1.2 General excitations

Now, we perform general excitations above the vacuum $\Omega_1^\uparrow \otimes \Omega_1^\downarrow$. We note

$$\mathbf{s}^\varepsilon = \mathbf{n}^\varepsilon + \mathbf{m}^\varepsilon = \mathbf{r}^\varepsilon + \bar{\mathbf{r}}^\varepsilon, \quad \mathbf{r}^\varepsilon = \text{rank}(\pi^\varepsilon), \quad \bar{\mathbf{r}}^\varepsilon = \text{rank}(\bar{\pi}^\varepsilon), \quad \varepsilon = \uparrow, \downarrow \quad (5.9)$$

$$\mathbf{s} = \mathbf{s}^\uparrow + \mathbf{s}^\downarrow. \quad (5.10)$$

We will have four types of excitations:

$$\left\{ \begin{array}{ll} \text{'unbarred' of type up:} & (a, \uparrow) \equiv j + 1, \quad 1 \leq j \leq \mathbf{r}^\uparrow - 1 \\ \text{'barred' of type up:} & (\bar{a}, \uparrow) \equiv j + 1, \quad \mathbf{r}^\uparrow \leq j \leq \mathbf{s}^\uparrow - 1 \\ \text{'unbarred' of type down:} & (a, \downarrow) \equiv j + 1 - \mathbf{s}^\uparrow, \quad \mathbf{s}^\uparrow \leq j \leq \mathbf{s}^\uparrow + \mathbf{r}^\downarrow - 2 \\ \text{'barred' of type down:} & (\bar{a}, \downarrow) \equiv j + 1 - \mathbf{s}^\uparrow, \quad \mathbf{s}^\uparrow + \mathbf{r}^\downarrow - 1 \leq j \leq \mathbf{s} - 2 \end{array} \right. \quad (5.11)$$

The set $\{(\bar{a}, \varepsilon)\}$ corresponds to the space $\overline{\mathcal{W}}^\varepsilon$, $\varepsilon = \uparrow, \downarrow$. The set $\{(a, \varepsilon)\}$ corresponds to the space \mathcal{W}^ε without the index 1 (that is associated to the vacuum): in the following, we will note this reduced space $\overset{\circ}{\mathcal{W}}^\varepsilon$.

One excitation For the states with one excitation, one has just to mimick what has been done in section 3.2.2.

$$\begin{aligned}\Phi_{a,\varepsilon}^1(p) &= \sum_{x=1}^L e^{ipx} |a, \varepsilon, x\rangle & a = 2, \dots, \text{rank}(\pi^\varepsilon) = \mathbf{r}^\varepsilon \\ \Phi_{\bar{a},\varepsilon}^1(p) &= \sum_{x=1}^L e^{ipx} |\bar{a}, \varepsilon, x\rangle & \bar{a} = \mathbf{r}^\varepsilon + 1, \dots, \mathbf{s}^\varepsilon = \mathbf{r}^\varepsilon + \bar{\mathbf{r}}^\varepsilon\end{aligned}, \quad \varepsilon = \uparrow, \downarrow \quad (5.12)$$

Through a direct calculation, it is easy to show that

$$\begin{aligned}t(0) \Phi_{\alpha,\varepsilon}^1(p) &= e^{ip} \Phi_{\alpha,\varepsilon}^1(p), \quad \alpha = a, \bar{a} \\ H \Phi_{a,\varepsilon}^1(p) &= UL \quad \text{and} \quad H \Phi_{\bar{a},\varepsilon}^1(p) = \left(2 \cos(p) + U(L-2)\right) \Phi_{\bar{a},\varepsilon}^1(p), \quad \varepsilon = \uparrow, \downarrow,\end{aligned} \quad (5.13)$$

if p obeys the Bethe ansatz equation (BAE)

$$e^{ipL} = 1 \quad (5.14)$$

Again, one can gather all these states into a single vector state. The labelling of excitations is done as explained in (5.11):

$$\{a = 2, \dots, \mathbf{r}^\uparrow; \bar{a} = \mathbf{r}^\uparrow + 1, \dots, \mathbf{s}^\uparrow = \mathbf{r}^\uparrow + \bar{\mathbf{r}}^\uparrow\} \rightarrow \{j = 1, \dots, \mathbf{s}^\uparrow - 1\} \quad (5.15)$$

$$\{a = 2, \dots, \mathbf{r}^\downarrow; \bar{a} = \mathbf{r}^\downarrow + 1, \dots, \mathbf{s}^\downarrow = \mathbf{r}^\downarrow + \bar{\mathbf{r}}^\downarrow\} \rightarrow \{j = \mathbf{s}^\downarrow, \dots, \mathbf{s} - 2\} \quad (5.16)$$

where the first $\mathbf{r}^\uparrow - 1$ indices are of type ‘ a, \uparrow ’, the next $\bar{\mathbf{r}}^\uparrow$ are of type ‘ \bar{a}, \uparrow ’, and so one. We introduce the elementary vectors $u_j \in \mathbb{C}^{\mathbf{s}-2}$ (with 1 at position j and 0 elsewhere) corresponding to the ‘small’ chain of the nested Bethe ansatz. The vector state reads:

$$\Phi^1(p) = \sum_{j=1}^{\mathbf{s}-2} \Phi_{j+1}^1(p) u_j = \sum_x e^{ipx} |x\rangle \quad \text{with} \quad |x\rangle = \sum_{j=1}^{\mathbf{s}-2} |j, x\rangle u_j \quad (5.17)$$

Note that in $|x\rangle$, $|j, x\rangle$ lies on the original ‘big’ chain (of length L), while u_j lies on a new ‘small’ chain (here of length 1). As in section 3.2.2, we ‘move’ the action of the transfer matrix and symmetry generators from the ‘big chain’ to the ‘small one’. We get

$$t(0) \Phi^1(p) = e^{ip} \Phi^1(p) \quad (5.18)$$

$$H \Phi^1(p) = \mathcal{D}(p) \Phi^1(p) = \left((2 \cos(p) - 2U) \bar{D} + UL \mathbb{I}_{\mathbf{s}-2}\right) \Phi^1(p) \quad (5.19)$$

$$\bar{D} = \text{diag}\left(\underbrace{0, \dots, 0}_{\mathbf{r}^\uparrow - 1}, \underbrace{1, \dots, 1}_{\bar{\mathbf{r}}^\uparrow}, \underbrace{0, \dots, 0}_{\mathbf{r}^\downarrow - 1}, \underbrace{1, \dots, 1}_{\bar{\mathbf{r}}^\downarrow}\right) \quad (5.20)$$

The matrix $\mathcal{D}(p)$ acts on the small chain (i.e. on the vectors u_j) while H was acting on the big chain (i.e. on the states $|j, x\rangle$). In the same way, the charges of the states are given by

$$\mathbb{M}_{j+1,j+1} \Phi^1(p) = E_{jj} \Phi^1(p), \quad j = 1, \dots, \mathbf{s} - 2 \quad (5.21)$$

$$\mathbb{M}_{11}^\varepsilon \Phi^1(p) = (L - D_\varepsilon) \Phi^1(p), \quad \varepsilon = \uparrow, \downarrow \quad (5.22)$$

$$D_\uparrow = \text{diag}\left(\underbrace{1, \dots, 1}_{\mathbf{s}^\uparrow - 1}, \underbrace{0, \dots, 0}_{\mathbf{s}^\downarrow - 1}\right) \quad \text{and} \quad D_\downarrow = \text{diag}\left(\underbrace{0, \dots, 0}_{\mathbf{s}^\uparrow - 1}, \underbrace{1, \dots, 1}_{\mathbf{s}^\downarrow - 1}\right) \quad (5.23)$$

where $E_{ij} \in \text{End}(\mathbb{C}^{s-2})$, $i, j > 1$, (the elementary matrix with 1 at position (i, j) and 0 elsewhere) acts on the small chain. It corresponds to the generator of the symmetry generator \mathbb{M}_{ij} acting on the big chain. $(L - \mathbb{M}_{11}^\varepsilon)$ corresponds to the excitation number for ε particles ($\varepsilon = \uparrow, \downarrow$).

Two excitations For more than one excitation, a new effect appears with respect to the XX models: there can be two excitations at the same site (provided there are of \uparrow and \downarrow type). To take it into account, we perform a change of basis on the states and define:

$$|x_1, x_2\rangle = \sum_{i=1}^{s-2} \sum_{j=1}^{s-2} |i+1, j+1; x_1, x_2\rangle u_i \otimes u_j \quad (5.24)$$

with the convention that

$$|i+1, j+1; x, x\rangle = \begin{cases} 0 & i, j \leq s^\uparrow - 1 \\ \frac{1}{2} |i+1, x\rangle \otimes |j+1, x\rangle & i \leq s^\uparrow - 1 < j \\ \frac{1}{2} |i+1, x\rangle \otimes |j+1, x\rangle & j \leq s^\uparrow - 1 < i \\ 0 & s^\uparrow - 1 < i, j \end{cases} \quad (5.25)$$

Then, the eigenstates are gathered into a vector

$$\Phi^2(p_1, p_2) = \sum_{i=1}^{s-2} \sum_{j=1}^{s-2} \Phi_{i,j}^2(p_1, p_2) u_i \otimes u_j \quad (5.26)$$

We have

$$\Phi^2(p_1, p_2) = \sum_{1 \leq x_1 \leq x_2 \leq L} \left\{ e^{i p \cdot x} \mathbb{I}_{s-2} \otimes \mathbb{I}_{s-2} + e^{i \gamma(p) \cdot x} P_{12} \mathcal{S}_{12}(p_1, p_2) \right\} |x_1, x_2\rangle \quad (5.27)$$

The scattering matrix is given by

$$\mathcal{S}_{12}(p_1, p_2) = \mathcal{S}_{12}^{X\uparrow}(p_1, p_2) + \mathcal{S}_{12}^{X\downarrow}(p_1, p_2) + \mathcal{S}_{12}^\uparrow(p_1, p_2) + \mathcal{S}_{12}^H(p_1, p_2) \quad (5.28)$$

$$\mathcal{S}_{12}^{X\varepsilon}(p_1, p_2) = e^{-ip_1} \overset{\circ}{\pi}^\varepsilon \otimes \bar{\pi}^\varepsilon + e^{ip_2} \bar{\pi}^\varepsilon \otimes \overset{\circ}{\pi}^\varepsilon - P_{12} \left(\overset{\circ}{\pi}^\varepsilon \otimes \overset{\circ}{\pi}^\varepsilon + \bar{\pi}^\varepsilon \otimes \bar{\pi}^\varepsilon \right), \quad \varepsilon = \uparrow, \downarrow \quad (5.29)$$

$$\mathcal{S}_{12}^\uparrow = \overset{\circ}{\pi}^\uparrow \otimes (\overset{\circ}{\pi}^\downarrow + \bar{\pi}^\downarrow) + (\overset{\circ}{\pi}^\downarrow + \bar{\pi}^\downarrow) \otimes \overset{\circ}{\pi}^\uparrow + \overset{\circ}{\pi}^\downarrow \otimes \bar{\pi}^\uparrow + \bar{\pi}^\uparrow \otimes \overset{\circ}{\pi}^\downarrow \quad (5.30)$$

$$\mathcal{S}_{12}^H(p_1, p_2) = \left(T(p_1, p_2) \mathbb{I}_{s-2} \otimes \mathbb{I}_{s-2} + R(p_1, p_2) P_{12} \right) \left(\bar{\pi}^\uparrow \otimes \bar{\pi}^\downarrow + \bar{\pi}^\downarrow \otimes \bar{\pi}^\uparrow \right) \quad (5.31)$$

$$T(p_1, p_2) = \frac{\sin(p_1) - \sin(p_2)}{\sin(p_1) - \sin(p_2) - 2iU} \quad (5.32)$$

$$R(p_1, p_2) = \frac{2iU}{\sin(p_1) - \sin(p_2) - 2iU} = T(p_1, p_2) - 1 \quad (5.33)$$

where $\overset{\circ}{\pi}^\varepsilon$ (resp. $\bar{\pi}^\varepsilon$) is the projector on $\overset{\circ}{\mathcal{W}}^\varepsilon$ (resp. $\overline{\mathcal{W}}^\varepsilon$), $\varepsilon = \uparrow, \downarrow$.

One recognizes in $\mathcal{S}_{12}^{X\varepsilon}(p_1, p_2)$ the scattering matrix of an XX model in the ' ε subsector' ($\varepsilon = \uparrow, \downarrow$). They correspond to the only part of \mathcal{S} which acts non trivially in the $\uparrow\uparrow$ and $\downarrow\downarrow$ sectors. The remaining part (acting in the $\downarrow\uparrow$ and $\uparrow\downarrow$ sectors) have been divided into a part acting only in the 'bar sector' (the \mathcal{S}^H matrix, of Heisenberg type) and the rest (the \mathcal{S}^\uparrow matrix).

Comparison with usual Hubbard model: the parts $\mathcal{S}_{12}^{X^\varepsilon}(p_1, p_2)$ and $\mathcal{S}_{12}^H(p_1, p_2)$ in the scattering matrix are just generalizations of the Hubbard scattering matrix to higher dimensional case. Note however the projections appearing in these scattering matrices, that are new w.r.t. the usual Hubbard model: we will comment on this point in section 5.2.4. The part \mathcal{S}^\uparrow is completely new: it introduces new physical effects that were not seen in Hubbard, due to the ‘small size’ of its vector space.

5.2 BAEs: a first account

Once the scattering matrix of the universal Hubbard model is known, the technique to obtain the transfer matrix eigenvalues and the BAEs is a priori known, see section 5.2.1 below. However, if the eigenvalues are easy to deduce, the determination of the precise form of the BAEs is a more delicate problem. Here, we compute them for some subsectors of the theory, leaving the determination of their complete form for a further publication.

5.2.1 M excitations

We consider a general state with M excitations, that divides into M^\uparrow excitations of type a in the ‘ \uparrow sector’, \bar{M}^\uparrow excitations of type \bar{a} in the ‘ \uparrow sector’, M^\downarrow excitations of type a in the ‘ \downarrow sector’, and \bar{M}^\downarrow excitations of type \bar{a} in the ‘ \downarrow sector’. The construction follows the line of section 3.2.4, with the noticeable exception that there can be \uparrow and \downarrow excitations at the same site. To take this fact into account, we introduce:

$$|\{j\}; \mathbf{x} \rangle = \begin{cases} \otimes_{m=1}^M |j_m; x_m \rangle & \text{if all } x_m \text{'s are different} \\ 0 & \text{if at least three } x_m \text{'s are equal} \end{cases} \quad (5.34)$$

$$|\{j\}; \mathbf{x} \rangle \Big|_{x_m=x_{m'}} = \begin{cases} 0 & \text{if } j, j' \leq \mathfrak{s}^\uparrow - 1 \\ \frac{1}{2} \otimes_{m=1}^M |j_m; x_m \rangle & \text{if } j \leq \mathfrak{s}^\uparrow - 1 < j' \\ \frac{1}{2} \otimes_{m=1}^M |j_m; x_m \rangle & \text{if } j' \leq \mathfrak{s}^\uparrow - 1 < j \\ 0 & \text{if } \mathfrak{s}^\uparrow - 1 < j, j' \end{cases} \quad (5.35)$$

Then, the BAEs take the form

$$e^{ip_j L} \Phi^M(\mathbf{p}) = \mathcal{S}_{j+1,j} \mathcal{S}_{j+2,j} \dots \mathcal{S}_{Mj} \mathcal{S}_{1j} \mathcal{S}_{2j} \dots \mathcal{S}_{j-1,j} \Phi^M(\mathbf{p}) \quad j = 1, \dots, M \quad (5.36)$$

In the following, we examine the BAEs in subsectors that are related to different types of excitations: the two XX-type subsectors, where excitations are only of type \uparrow or only of type \downarrow ; the ‘unbarred subsector’, where excitations are only of type ‘unbarred’ (\uparrow or \downarrow), and the Hubbard-type subsector, where excitations are only of type ‘bar’ (\uparrow or \downarrow).

5.2.2 BAEs for the XX-type subsector

We introduce the projectors on the ‘ \uparrow sector’ and ‘ \downarrow sector’

$$\Pi^\varepsilon = \overset{\circ}{\pi}^\varepsilon + \bar{\pi}^\varepsilon, \quad \varepsilon = \uparrow, \downarrow. \quad (5.37)$$

It is easy to see that

$$\Pi^\varepsilon \otimes \Pi^\varepsilon \mathcal{S}_{12}(p_1, p_2) = \mathcal{S}_{12}^{X^\varepsilon}(p_1, p_2) \Pi^\varepsilon \otimes \Pi^\varepsilon \quad \varepsilon = \uparrow, \downarrow \quad (5.38)$$

so that multiplying the BAEs from the left, by $(\Pi^\varepsilon)^{\otimes M}$, one recovers the BAEs of the XX models:

$$\begin{aligned} e^{ip_j L} \Phi_\uparrow^M(\mathbf{p}) &= \mathcal{S}_{j+1,j}^{X\uparrow} \mathcal{S}_{j+2,j}^{X\uparrow} \dots \mathcal{S}_{Mj}^{X\uparrow} \mathcal{S}_{1j}^{X\uparrow} \mathcal{S}_{2j}^{X\uparrow} \dots \mathcal{S}_{j-1,j}^{X\uparrow} \Phi_\uparrow^M(\mathbf{p}) \\ j &= 1, \dots, M \quad ; \quad M^\downarrow = \bar{M}^\downarrow = 0 \end{aligned} \quad (5.39)$$

$$\begin{aligned} e^{ip_j L} \Phi_\downarrow^M(\mathbf{p}) &= \mathcal{S}_{j+1,j}^{X\downarrow} \mathcal{S}_{j+2,j}^{X\downarrow} \dots \mathcal{S}_{Mj}^{X\downarrow} \mathcal{S}_{1j}^{X\downarrow} \mathcal{S}_{2j}^{X\downarrow} \dots \mathcal{S}_{j-1,j}^{X\downarrow} \Phi_\downarrow^M(\mathbf{p}) \\ j &= 1, \dots, M \quad ; \quad M^\uparrow = \bar{M}^\uparrow = 0 \end{aligned} \quad (5.40)$$

$$\Phi_\varepsilon^M(\mathbf{p}) = \underbrace{(\Pi^\varepsilon \otimes \Pi^\varepsilon \otimes \dots \otimes \Pi^\varepsilon)}_M \Phi^M(\mathbf{p}), \quad \varepsilon = \uparrow, \downarrow \quad (5.41)$$

These BAEs corresponds to subsectors where excitations of only \uparrow or only \downarrow types are considered. They are of the same form that the XX models:

$$\exp(i q_n (L - M'')) = (-1)^{M'-1} (\omega_{M'})^n, \quad n = 1, \dots, M' \quad \text{with} \quad (\omega_{M'})^{M'} = 1 \quad (5.42)$$

$$\exp(i L \bar{q}_n) = (\omega_{M''})^n \exp(-i |\mathbf{q}|), \quad n = 1, \dots, M'' \quad \text{with} \quad (\omega_{M''})^{M''} = 1. \quad (5.43)$$

5.2.3 BAEs for the ‘unbarred sector’

We consider the ‘unbarred subsector’, i.e. states with unbarred excitations (of type \uparrow or \downarrow) only. The corresponding projector is

$$\Pi = \overset{\circ}{\pi}^\uparrow + \overset{\circ}{\pi}^\downarrow \quad ; \quad \Pi_{12} = \Pi \otimes \Pi \quad ; \quad \Pi_{1\dots M} = \underbrace{\Pi \otimes \Pi \otimes \dots \otimes \Pi \otimes \Pi}_M \quad (5.44)$$

From the property

$$\Pi_{1\dots M} \mathcal{S}_{12}(p_1, p_2) = \mathcal{S}_{12}^{un}(p_1, p_2) \Pi_{1\dots M} \quad (5.45)$$

$$\mathcal{S}_{12}^{un}(p_1, p_2) = \sum_{\varepsilon=\uparrow, \downarrow} \left\{ \overset{\circ}{\pi}^\varepsilon \otimes \overset{\circ}{\pi}^{-\varepsilon} - P_{12} \left(\overset{\circ}{\pi}^\varepsilon \otimes \overset{\circ}{\pi}^\varepsilon \right) \right\} \quad (5.46)$$

it is easy to see that the calculation is very similar to the XX case, with $\overset{\circ}{\pi}^\uparrow, \overset{\circ}{\pi}^\downarrow$ playing the role of $\overset{\circ}{\pi}, \bar{\pi}$ of section 3.2.4. Using the same notations, one gets

$$\begin{aligned} \mathcal{S}_{j+1,j}^{un} \mathcal{S}_{j+2,j}^{un} \dots \mathcal{S}_{Mj}^{un} \mathcal{S}_{1j}^{un} \mathcal{S}_{2j}^{un} \dots \mathcal{S}_{j-1,j}^{un} &= \sum_{n=0}^M \sum_{\{j\}_n^{\prec} \oplus \{k\}} (-1)^n P_{jj_1} P_{jj_2} \dots P_{jj_n} \left\{ \right. \\ &\quad \left. \overset{\circ}{\pi}_j^\uparrow \overset{\circ}{\pi}_{j_1}^\uparrow \dots \overset{\circ}{\pi}_{j_n}^\uparrow \overset{\circ}{\pi}_{k_1}^\downarrow \dots \overset{\circ}{\pi}_{k_{M-1-n}}^\downarrow + \overset{\circ}{\pi}_j^\downarrow \overset{\circ}{\pi}_{j_1}^\downarrow \dots \overset{\circ}{\pi}_{j_n}^\downarrow \overset{\circ}{\pi}_{k_1}^\uparrow \dots \overset{\circ}{\pi}_{k_{M-1-n}}^\uparrow \right\} \end{aligned}$$

Since in this sector the BAEs take the form

$$\begin{aligned} e^{ip_j L} \Phi_{un}^M(\mathbf{p}) &= \mathcal{S}_{j+1,j}^{un} \mathcal{S}_{j+2,j}^{un} \dots \mathcal{S}_{Mj}^{un} \mathcal{S}_{1j}^{un} \mathcal{S}_{2j}^{un} \dots \mathcal{S}_{j-1,j}^{un} \Phi_{un}^M(\mathbf{p}) \\ \Phi_{un}^M(\mathbf{p}) &= \Pi_{1\dots M} \Phi^M(\mathbf{p}) \end{aligned}$$

we obtain

$$\exp\left(i q_n L\right) = (-1)^{M^\uparrow-1} (\omega_{M^\uparrow})^n, \quad n = 1, 2, \dots, M^\uparrow \quad \text{with} \quad (\omega_{M^\uparrow})^{M^\uparrow} = 1, \quad (5.47)$$

$$\exp\left(i q'_n L\right) = (-1)^{M^\downarrow-1} (\omega_{M^\downarrow})^n, \quad n = 1, 2, \dots, M^\downarrow \quad \text{with} \quad (\omega_{M^\downarrow})^{M^\downarrow} = 1 \quad (5.48)$$

where M^\uparrow is the number of \uparrow excitations, and $M^\downarrow = M - M^\uparrow$ is the number of \downarrow excitations. We noted q_n , $n = 1, 2, \dots, M^\uparrow$, the momenta of the \uparrow excitations and q'_n , $n = 1, 2, \dots, M^\downarrow$, the momenta of the \downarrow excitations.

Remark that the two series of BAEs (5.47) and (5.48) are decoupled, and correspond to the ‘unbarred sector’ of each of the underlying XX models. They can be obtained separately using the projectors

$$\Pi_{1\dots M}^\uparrow = \underbrace{\overset{\circ}{\pi}^\uparrow \otimes \overset{\circ}{\pi}^\uparrow \otimes \dots \otimes \overset{\circ}{\pi}^\uparrow \otimes \overset{\circ}{\pi}^\uparrow}_M \quad \text{or} \quad \Pi_{1\dots M}^\downarrow = \underbrace{\overset{\circ}{\pi}^\downarrow \otimes \overset{\circ}{\pi}^\downarrow \otimes \dots \otimes \overset{\circ}{\pi}^\downarrow \otimes \overset{\circ}{\pi}^\downarrow}_M, \quad (5.49)$$

but the present calculation shows that they are complete in this subsector.

5.2.4 The ‘bar subsector’

Following the same lines as in the previous sections, one can consider the ‘bar subsector’, i.e. states with \bar{a}_\uparrow and \bar{a}_\downarrow excitations only. The corresponding projector is

$$\bar{\Pi} = \bar{\pi}^\uparrow + \bar{\pi}^\downarrow \quad \text{and} \quad \bar{\Pi}_{1\dots M} = \underbrace{\bar{\Pi} \otimes \bar{\Pi} \otimes \dots \otimes \bar{\Pi}}_M. \quad (5.50)$$

In that case, one has

$$\begin{aligned} \bar{\Pi}_{1\dots M} \mathcal{S}_{12}(p_1, p_2) &= \mathcal{S}_{12}(p_1, p_2) \bar{\Pi}_{1\dots M} \equiv \bar{\mathcal{S}}_{12}(p_1, p_2) \bar{\Pi}_{1\dots M} \\ \bar{\mathcal{S}}_{12}(p_1, p_2) &= \left(T(p_1, p_2) \mathbb{I}_{\bar{s}-2} \otimes \mathbb{I}_{\bar{s}-2} + R(p_1, p_2) P_{12} \right) \left(\bar{\pi}^\uparrow \otimes \bar{\pi}^\downarrow + \bar{\pi}^\downarrow \otimes \bar{\pi}^\uparrow \right) \\ &\quad - P_{12} \left(\bar{\pi}^\uparrow \otimes \bar{\pi}^\uparrow + \bar{\pi}^\downarrow \otimes \bar{\pi}^\downarrow \right) \end{aligned} \quad (5.51)$$

One could be tempted to recognize in $\bar{\mathcal{S}}_{12}$, the scattering matrix of a generalized XXX model. Indeed, specifying the \uparrow or \downarrow type only (whatever the indices $\bar{a}, \bar{b}, \bar{c}, \dots$ are), one gets on a state with two excitations:

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\uparrow\uparrow'> = -|\uparrow\uparrow'> \quad (5.52)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\uparrow\downarrow> = T(p_1, p_2) |\uparrow\downarrow> + R(p_1, p_2) |\downarrow\uparrow> \quad (5.53)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\downarrow\uparrow> = R(p_1, p_2) |\uparrow\downarrow> + T(p_1, p_2) |\downarrow\uparrow> \quad (5.54)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\downarrow\downarrow'> = -|\downarrow\downarrow'> \quad (5.55)$$

In the case of Hubbard model, where there is only one type of \uparrow excitation and one type of \downarrow excitation, one has exactly the scattering matrix of the XXX models. This allowed the calculation of BAEs of the Hubbard model. However, if there is more than one type of \uparrow or

\downarrow excitation, this is not the case anymore. For instance, for two types of \uparrow excitations (say \bar{a} and \bar{b}), eq. (5.52) corresponds to

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{a} \uparrow; \bar{a} \uparrow' \rangle = -|\bar{a} \uparrow'; \bar{a} \uparrow \rangle \quad (5.56)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{a} \uparrow; \bar{b} \uparrow' \rangle = -|\bar{b} \uparrow'; \bar{a} \uparrow \rangle \quad (5.57)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{b} \uparrow; \bar{a} \uparrow' \rangle = -|\bar{a} \uparrow'; \bar{b} \uparrow \rangle \quad (5.58)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{b} \uparrow; \bar{b} \uparrow' \rangle = -|\bar{b} \uparrow'; \bar{b} \uparrow \rangle \quad (5.59)$$

while a generalized XXX model would act as

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{a} \uparrow; \bar{a} \uparrow' \rangle = -|\bar{a} \uparrow'; \bar{a} \uparrow \rangle \quad (5.60)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{a} \uparrow; \bar{b} \uparrow \rangle = T(p_1, p_2) |\bar{a} \uparrow; \bar{b} \uparrow \rangle + R(p_1, p_2) |\bar{b} \uparrow; \bar{a} \uparrow \rangle \quad (5.61)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{b} \uparrow; \bar{a} \uparrow \rangle = T(p_1, p_2) |\bar{b} \uparrow; \bar{a} \uparrow \rangle + R(p_1, p_2) |\bar{a} \uparrow; \bar{b} \uparrow \rangle \quad (5.62)$$

$$\bar{\mathcal{S}}_{12}(p_1, p_2) |\bar{b} \uparrow; \bar{b} \uparrow' \rangle = -|\bar{b} \uparrow'; \bar{b} \uparrow \rangle \quad (5.63)$$

This difference just prevents to perform a nesting in the same way it is done for Hubbard. Note that $\bar{\mathcal{S}}_{12}$ still obeys

$$\bar{\mathcal{S}}_{12}(p, p) = -P_{12} \quad (5.64)$$

so that one can define an integrable spin chain associated to the nesting in the usual way. However, the exact form of the BAEs for this new chain is not known yet. The same is true for the general BAEs of the universal Hubbard model. We will come back on this point in a further work [23].

6 Perturbative expansion of the Hubbard-like Hamiltonian

We expand the Hamiltonian (4.13) and (4.14) in the inverse coupling $\frac{1}{U}$. That expansion has been used in [13] to match the $SU(2)$ dilatation operator with the effective Hamiltonian of the Hubbard model. The system was taken at half-filling to guarantee the required spin chain behaviour.

Being ultralocal, the potential term $U \sum_j C_j^\uparrow C_j^\downarrow$ is separately diagonalizable on each site with eigenvalues $\pm U$. Indeed, they are obtained from the property $C^2 = \mathbb{I}$ (2.6).

The ground state has eigenvalue $-LU$ and can be obtained if the condition $C_j^\uparrow C_j^\downarrow = -1$ is realised on each site. This is equivalent to demand eigenvalue 1 on each site for the (one-site) projector

$$\frac{1 - C_j^\uparrow C_j^\downarrow}{2} = \left(\frac{1 - C_j^\uparrow C_j^\downarrow}{2} \right)^2 = \pi_j^\uparrow + \pi_j^\downarrow - 2\pi_j^\uparrow \pi_j^\downarrow = (\pi_j^\uparrow - \pi_j^\downarrow)^2 \quad (6.1)$$

or the global projector

$$\Pi_0 = \prod_j (\pi_j^\uparrow - \pi_j^\downarrow)^2 = \prod_j (\bar{\pi}_j^\uparrow - \bar{\pi}_j^\downarrow)^2 = \Pi_0^2. \quad (6.2)$$

We observe that it projects on the subspace where, on each site, one and only one projector among $\bar{\pi}_j^\uparrow, \bar{\pi}_j^\downarrow$ has nonzero action. This means that only the following subspaces survive

$$\Pi_0 : \quad \mathcal{W}_j^\uparrow \otimes \bar{\mathcal{W}}_j^\downarrow \quad \text{or} \quad \bar{\mathcal{W}}_j^\uparrow \otimes \mathcal{W}_j^\downarrow \quad (6.3)$$

Namely we demand that on each site there is one barred particle: double or empty occupancies of barred particles are prohibited. This may be possible only if the system has precisely L barred particles out of the $2L$ permitted ones. We say that the system is *half-filled* and we will assume this condition to perform the perturbative calculations.

It is useful to compare with the ordinary Hubbard model, where the algebra is realised in terms of fermionic oscillators $c_{\sigma,j}, c_{\sigma,j}^\dagger$, satisfying $\{c_{\sigma,j}, c_{\sigma,j}^\dagger\} = 1$ ($\sigma = \uparrow, \downarrow$). There, the projector $\bar{\pi}_j^\sigma$ is equal to the number operator $n_{\sigma,j} = c_{\sigma,j}^\dagger c_{\sigma,j}$, so in the present general formalism, a vector of $\bar{\mathcal{W}}^\sigma$ corresponds to an electron and a vector of \mathcal{W}^σ corresponds to a vacancy (in the Hubbard model we have $\pi_j^\sigma = 1 - n_{\sigma,j}$).

We follow the method introduced by Klein and Seitz [20]. With reference to the Hamiltonian (4.14) and in complete analogy with previous cases [16], we define a hopping operator by

$$X_{ij} = P_{ij}^\uparrow \pi_i^\uparrow \bar{\pi}_j^\uparrow + P_{ij}^\downarrow \pi_i^\downarrow \bar{\pi}_j^\downarrow = \bar{\pi}_i^\uparrow \pi_j^\uparrow P_{ij}^\uparrow \pi_i^\uparrow \bar{\pi}_j^\uparrow + \bar{\pi}_i^\downarrow \pi_j^\downarrow P_{ij}^\downarrow \pi_i^\downarrow \bar{\pi}_j^\downarrow$$

Intuitively we associate its action to the move of a barred particle from site j to site i . We define hermitian conjugation as super-transposition (because our operators are real) so that we have

$$X_{ij}^\dagger = X_{ji} \quad (6.4)$$

The two-sites Hamiltonian (4.14) takes the form

$$H_{j,j+1} = X_{j,j+1} + X_{j+1,j} + U C_j^\uparrow C_j^\downarrow \quad (6.5)$$

and is obviously self-adjoint. The perturbing term is

$$T = \sum_j (X_{j,j+1} + X_{j+1,j}). \quad (6.6)$$

The action of X_{ij} on the vector spaces defined in (2.1) is easily described by observing the projectors in the second line of (6.4). We initially focus on the effect on site i :

$$\begin{aligned} 1) \quad & \mathcal{W}_i^\uparrow \otimes \mathcal{W}_i^\downarrow \xrightarrow{X_{ij}} \bar{\mathcal{W}}_i^\uparrow \otimes \mathcal{W}_i^\downarrow \oplus \mathcal{W}_i^\uparrow \otimes \bar{\mathcal{W}}_i^\downarrow \\ 2) \quad & \bar{\mathcal{W}}_i^\uparrow \otimes \bar{\mathcal{W}}_i^\downarrow \xrightarrow{X_{ij}} 0 \\ 3) \quad & \mathcal{W}_i^\uparrow \otimes \bar{\mathcal{W}}_i^\downarrow \xrightarrow{X_{ij}} \bar{\mathcal{W}}_i^\uparrow \otimes \bar{\mathcal{W}}_i^\downarrow \\ 4) \quad & \bar{\mathcal{W}}_i^\uparrow \otimes \mathcal{W}_i^\downarrow \xrightarrow{X_{ij}} \bar{\mathcal{W}}_i^\uparrow \otimes \mathcal{W}_i^\downarrow \end{aligned} \quad (6.7)$$

On site j the description is the complementary one, namely:

$$\begin{aligned}
1) \quad & \mathcal{W}_j^\uparrow \otimes \mathcal{W}_j^\downarrow \xrightarrow{X_{ij}} 0 \\
2) \quad & \overline{\mathcal{W}}_j^\uparrow \otimes \overline{\mathcal{W}}_j^\downarrow \xrightarrow{X_{ij}} \mathcal{W}_j^\uparrow \otimes \overline{\mathcal{W}}_j^\downarrow \oplus \overline{\mathcal{W}}_j^\uparrow \otimes \mathcal{W}_j^\downarrow \\
3) \quad & \mathcal{W}_j^\uparrow \otimes \overline{\mathcal{W}}_j^\downarrow \xrightarrow{X_{ij}} \mathcal{W}_j^\uparrow \otimes \mathcal{W}_j^\downarrow \\
4) \quad & \overline{\mathcal{W}}_j^\uparrow \otimes \mathcal{W}_j^\downarrow \xrightarrow{X_{ij}} \mathcal{W}_j^\uparrow \otimes \mathcal{W}_j^\downarrow
\end{aligned} \tag{6.8}$$

It is clear that the domain and the codomain of X_{ij} are always disjoint and only a “two-fold” action can make them the same. We state this as a theorem.

Theorem 6.1 *Given a site i and an initial configuration choosen among the four listed in (6.7), the product of an odd number of operators X_{ij} or X_{ji} acting on i with possibly different j cannot return to the same initial configuration.*

Proof: The proof is a trivial application of the rules in (6.7 and 6.8) ■

A number of corollaries follow from it and from the action of the projector Π_0 :

$$\begin{aligned}
X_{ij}^2 \Pi_0 &= 0 \\
(1 - \Pi_0) X_{ij} X_{ji} \Pi_0 &= 0 \\
X_{j-1,j} X_{j+1,j} \Pi_0 &= 0 \\
\Pi_0 T^n \Pi_0 &= 0 \quad \text{if } n = \text{odd} \text{ and } L > n
\end{aligned} \tag{6.9}$$

The condition $L > n$ in (6.9) is extremely important. If it is removed, new terms known as demi-wrapping $L = n$ and wrapping $L < n$ behaviours occur because the perturbative interaction (6.6) circulates all around the periodic chain. These terms will not be evaluated here.

According to Klein and Seitz [20], the effective Hamiltonian for the states originated from Π_0 is

$$H_{\text{eff}} = \frac{1}{U} H_{\text{eff}}^{(2)} + \frac{1}{U^3} H_{\text{eff}}^{(4)} + \dots \tag{6.10}$$

where

$$S = (1 - \Pi_0) \frac{1}{E_0 - H_0} (1 - \Pi_0) \tag{6.11}$$

$$H_{\text{eff}}^{(2)} = \Pi_0 T S T \Pi_0 = -\frac{1}{4} \Pi_0 T^2 \Pi_0 \tag{6.12}$$

$$H_{\text{eff}}^{(4)} = \frac{1}{16} \Pi_0 T^2 S T^2 \Pi_0 + \frac{1}{64} \Pi_0 T^2 \Pi_0 T^2 \Pi_0 \tag{6.13}$$

These expressions can be worked out following Klein and Seitz [20], with a long but simple calculation that is not shown here, using the properties (6.9).

We point out another property that allows to simplify the expressions: the projector Π_0 makes redundant one among the barred and the non-barred projectors. Indeed, we can write the identity on site j as $\pi_j^\sigma + \bar{\pi}_j^\sigma = 1$, therefore

$$\pi_j^\sigma \Pi_0 = \pi_j^\sigma (\pi_j^{-\sigma} + \bar{\pi}_j^{-\sigma}) \Pi_0 = \pi_j^\sigma \bar{\pi}_j^{-\sigma} \Pi_0 = \bar{\pi}_j^{-\sigma} \Pi_0 \quad (6.14)$$

and we can write all the expressions using only the non-barred projectors.

6.1 Second order Hamiltonian

A direct calculation shows that for $L > 2$ the second order effective Hamiltonian is

$$\begin{aligned} H_{\text{eff}}^{(2)} &= \sum_j H_{\text{eff } j, j+1}^{(2)} \\ &= \Pi_0 \left(2 \sum_j (1 + P_{j, j+1}^\uparrow P_{j, j+1}^\downarrow) (\pi_j^\uparrow \pi_{j+1}^\downarrow + \pi_j^\downarrow \pi_{j+1}^\uparrow) \right) \Pi_0 \end{aligned} \quad (6.15)$$

The structure of the two-sites Hamiltonian $H_{\text{eff } 1, 2}^{(2)}$ can be described in the following way. The projector Π_0 allows states of the form (6.3), so we start observing that the Hamiltonian vanishes on states of the following form

$$\mathcal{W}_1^\uparrow \otimes \bar{\mathcal{W}}_1^\downarrow \otimes \mathcal{W}_2^\uparrow \otimes \bar{\mathcal{W}}_2^\downarrow \quad \text{or} \quad \bar{\mathcal{W}}_1^\uparrow \otimes \mathcal{W}_1^\downarrow \otimes \bar{\mathcal{W}}_2^\uparrow \otimes \mathcal{W}_2^\downarrow \quad (6.16)$$

because the projectors in (6.15) require orthogonal subspaces on different sites for the same type (e.g. up) of vectors. For example, this means that a state $v_1^\uparrow \otimes \bar{v}_1^\downarrow \otimes v_2^\uparrow \otimes \bar{v}_2^\downarrow$ is killed by the two-sites Hamiltonian. We are left with states of the form

$$\mathcal{W}_1^\uparrow \otimes \bar{\mathcal{W}}_1^\downarrow \otimes \bar{\mathcal{W}}_2^\uparrow \otimes \mathcal{W}_2^\downarrow \quad \text{or} \quad \bar{\mathcal{W}}_1^\uparrow \otimes \mathcal{W}_1^\downarrow \otimes \mathcal{W}_2^\uparrow \otimes \bar{\mathcal{W}}_2^\downarrow \quad (6.17)$$

on which the parenthesis of projectors $(\pi_j^\uparrow \pi_{j+1}^\downarrow + \pi_j^\downarrow \pi_{j+1}^\uparrow)$ acts as the identity. A state in one of the spaces (6.17) is respectively of the form

$$v_1 \otimes \bar{w}_1 \otimes \bar{v}_2 \otimes w_2 \quad , \quad \bar{v}_1 \otimes w_1 \otimes v_2 \otimes \bar{w}_2 \quad (6.18)$$

on which we have respectively

$$\begin{aligned} (1 + P_{j, j+1}^\uparrow P_{j, j+1}^\downarrow) v_1 \otimes \bar{w}_1 \otimes \bar{v}_2 \otimes w_2 &= v_1 \otimes \bar{w}_1 \otimes \bar{v}_2 \otimes w_2 + (-1)^{([v]+[\bar{w}])([\bar{v}]+[w])} \bar{v}_1 \otimes w_1 \otimes v_2 \otimes \bar{w}_2 \\ (1 + P_{j, j+1}^\uparrow P_{j, j+1}^\downarrow) \bar{v}_1 \otimes w_1 \otimes v_2 \otimes \bar{w}_2 &= \bar{v}_1 \otimes w_1 \otimes v_2 \otimes \bar{w}_2 + (-1)^{([v]+[\bar{w}])([\bar{v}]+[w])} v_1 \otimes \bar{w}_1 \otimes \bar{v}_2 \otimes w_2 \end{aligned} \quad (6.19)$$

In matricial form, the two-sites Hamiltonian has one of the two block-diagonal structures

$$B_- = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad B_+ = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \quad (6.20)$$

all other entries being zero [16]. Both the blocks have eigenvalues 0 and 2. The multiplicity depends on the actual model under examination. In the Hubbard model, the effective Hamiltonian acting on the singly occupied states reduces to the block B_- only

$$H_{\text{eff},j,j+1}^{(2)} = 1 - 4 \mathbf{S}_j \cdot \mathbf{S}_{j+1} = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.21)$$

where $\mathbf{S}_j = (S^x, S^y, S^z)$ on site j are the spin vectors of the Heisenberg model.

6.2 Fourth order Hamiltonian

The fourth order Hamiltonian

$$\begin{aligned} H_{\text{eff}}^{(4)} &= \sum_j H_{\text{eff},j,j+1,j+2}^{(4)} = \\ &= \frac{1}{32} \sum_j \left[(1 + 2P_{j,j+1}^\uparrow P_{j,j+1}^\downarrow + P_{j+1,j+2}^\uparrow P_{j+1,j+2}^\downarrow P_{j,j+1}^\uparrow P_{j,j+1}^\downarrow) (\pi_j^\downarrow \pi_{j+1}^\uparrow \pi_{j+2}^\uparrow + \pi_j^\uparrow \pi_{j+1}^\downarrow \pi_{j+2}^\downarrow) + \right. \\ &\quad + (1 + 2P_{j+1,j+2}^\uparrow P_{j+1,j+2}^\downarrow + P_{j,j+1}^\uparrow P_{j,j+1}^\downarrow P_{j+1,j+2}^\uparrow P_{j+1,j+2}^\downarrow) (\pi_j^\downarrow \pi_{j+1}^\downarrow \pi_{j+2}^\uparrow + \pi_j^\uparrow \pi_{j+1}^\uparrow \pi_{j+2}^\downarrow) + \\ &\quad \left. + 2(2 + P_{j,j+1}^\uparrow P_{j,j+1}^\downarrow + P_{j+1,j+2}^\uparrow P_{j+1,j+2}^\downarrow) (\pi_j^\uparrow \pi_{j+1}^\downarrow \pi_{j+2}^\uparrow + \pi_j^\downarrow \pi_{j+1}^\uparrow \pi_{j+2}^\downarrow) \right] \Pi_0 \end{aligned} \quad (6.22)$$

is composed by a three-sites Hamiltonian density. It acts generically on states $\pi_j^{\varepsilon_1} \pi_{j+1}^{\varepsilon_2} \pi_{j+2}^{\varepsilon_3}$, where $\varepsilon = \pm 1$ indicates respectively \uparrow and \downarrow . It cannot mix states with different values of $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$; if that sum is ± 3 , its action is zero. The other two possible values are ± 1 , on which it acts independently.

The second order Hamiltonian can be put in a three-sites density form by averaging on neighboring sites

$$\frac{1}{2} (H_{\text{eff},j,j+1}^{(2)} + H_{\text{eff},j+1,j+2}^{(2)}) \quad (6.23)$$

therefore we can evaluate the eigenvalues of a three-sites Hamiltonian formed by

$$\mathcal{H}_{\text{eff},j,j+1,j+2} = \frac{1}{2} (H_{\text{eff},j,j+1}^{(2)} + H_{\text{eff},j+1,j+2}^{(2)}) + \frac{1}{U^2} H_{\text{eff},j,j+1,j+2}^{(4)} \quad (6.24)$$

by action on states as in (6.19). The possible eigenvalues are (up to corrections of order U^{-4}) the same in all cases

$$\text{Eigen}(\mathcal{H}_{\text{eff},j,j+1,j+2}) = \left\{ 0, 1, 3 \left(1 + \frac{1}{16U^2} \right) \right\} \quad (6.25)$$

with multiplicities that depend on the specific model under consideration.

7 Conclusion

We have defined in a very general way Hubbard models with arbitrary symmetry. The basic ingredients are a vector space, which defines the representation space on each site, and two projectors, which separate the particles into two classes that behave in a different way. The scattering matrix, as well as the energies, have been computed on very general ground. The general form of Bethe ansatz equations remains to be computed, although some of them are given here. Of course applications of these models to condensed matter physics and/or AdS/CFT correspondence are of first importance. In this regard, we have given in appendices a general procedure to include a Aharonov-Bohm phase and some hints towards the definition of integrable bosonic Hubbard models. This latter feature is especially crucial, in particular for applications in string theory regarding the AdS/CFT correspondence. Indeed, dealing with models with $psu(4|4)$ symmetry, one is lead to consider some subsectors of the theory containing both fermionic and bosonic particles. The study of condensed matter models with bosonic content on an integrable point of view also requires progress in this direction. Finally, let us stress that the input of boundaries, which play a great role in condensed matter models, may be a worthwhile extension of this work, see e.g. [24] for the one-dimensional Hubbard model with integrable boundaries.

A Towards integrable bosonic Hubbard models

Our construction can in principle be applied to infinite dimensional vector spaces, leading to possible bosonic integrable Hubbard models. However, for such a purpose, one needs to construct a trace operator on the space \mathcal{V} , a task that is not guaranteed when \mathcal{V} is infinite dimensional. To simplify the presentation, we work on XX models, but the procedure leading to Hubbard models can be applied in the same way we did for finite-dimensional vector spaces.

A.1 R -matrices associated to Fock space

To illustrate the problems encountered in the infinite dimensional case, we focus on the case where \mathcal{V} is the Fock space \mathcal{F} , based on oscillators (b, b^\dagger) with $[b, b^\dagger] = 1$. We denote by $|0\rangle$ the Fock vacuum, and $|n\rangle = (b^\dagger)^n |0\rangle$, $n = 0, 1, 2, \dots$, and by $\hat{N} = b^\dagger b$ the number operator:

$$\hat{N} |n\rangle = n |n\rangle. \quad (\text{A.1})$$

The dual vectors are noted $\langle n|$:

$$\langle m| \cdot |n\rangle \equiv \langle m|n\rangle = \delta_{n,m}. \quad (\text{A.2})$$

We also introduce the subsets

$$\mathcal{F}_N = \left\{ |n\rangle, n = 0, 1, 2, \dots, N \right\} \subset \mathcal{F}, \quad N = 0, 1, 2, \dots \quad (\text{A.3})$$

The permutation operator is given by

$$P_{12} = \sum_{n,m=0}^{\infty} |n\rangle \langle m| \otimes |m\rangle \langle n| \quad (\text{A.4})$$

We present two examples of projectors.

Even-odd projectors: One can choose as projectors

$$\pi^{ev} = \frac{1}{2}(1 + (-1)^{\hat{N}}) \quad \text{and} \quad \bar{\pi}^{ev} = \mathbb{I} - \pi^{ev} \equiv \pi^{odd} = \frac{1}{2}(1 - (-1)^{\hat{N}}), \quad (\text{A.5})$$

which project on even and odd particle number eigenspaces \mathcal{F}_{ev} and \mathcal{F}_{odd} . They obviously commute with \hat{N} . The parity operator is $C = (-1)^{\hat{N}}$. Then, the corresponding XX R -matrix reads

$$R(\lambda) = \cos\left(\frac{\lambda}{2}\right) \left\{ \cos\left(\frac{\lambda}{2}\right) P_{12} + \sin\left(\frac{\lambda}{2}\right) \mathbb{I} \otimes \mathbb{I} \right\} - \sin\left(\frac{\lambda}{2}\right) (-1)^{\hat{N}_1 + \hat{N}_2} \left\{ \sin\left(\frac{\lambda}{2}\right) P_{12} + \cos\left(\frac{\lambda}{2}\right) \mathbb{I} \otimes \mathbb{I} \right\} \quad (\text{A.6})$$

where $\hat{N}_1 = \hat{N} \otimes \mathbb{I}$ and $\hat{N}_2 = \mathbb{I} \otimes \hat{N}$.

Small modes projector: For any number $\ell \in \mathbb{Z}_+$, one can also take as projectors

$$\pi^{\leq \ell} = \sum_{n=0}^{\ell} |n\rangle \langle n| \quad \text{and} \quad \bar{\pi}^{\leq \ell} = \mathbb{I} - \pi^{\leq \ell} \equiv \pi^{> \ell} = \sum_{n>\ell} |n\rangle \langle n|. \quad (\text{A.7})$$

They also commute with \hat{N} . The parity operator reads

$$C = \sum_{n=0}^{\ell} |n\rangle \langle n| - \sum_{n=\ell+1}^{\infty} |n\rangle \langle n|. \quad (\text{A.8})$$

Properties of the R -matrix: The different types of projectors lead to different types of R -matrices. From the general treatment done in section 2.1, one already knows that these R -matrices obey the theorem 2.1, in particular the Yang-Baxter equation. Then, one directly constructs a monodromy matrix

$$\mathcal{L}_{0<1\dots L>}(\lambda) = R_{01}(\lambda) R_{02}(\lambda) \cdots R_{0L}(\lambda) \quad (\text{A.9})$$

which obeys the relation

$$R_{00'}(\lambda - \mu) \mathcal{L}_{0<1\dots L>}(\lambda) \mathcal{L}_{0'<1\dots L>}(\mu) = \mathcal{L}_{0'<1\dots L>}(\mu) \mathcal{L}_{0<1\dots L>}(\lambda) R_{00'}(\lambda - \mu). \quad (\text{A.10})$$

However, to get a transfer matrix leading to an integrable model, one first needs to define the trace operator: we discuss it in the next section.

We recall that the same construction is valid for the Hubbard R -matrices that one could build by coupling two XX models, as done in section 4.1.1.

A.2 Trace operator and transfer matrix

One can define a trace operator for operators \mathcal{O} such that $[\mathcal{O}, \hat{N}] = 0$ in the following way. Since such operators preserve the subsets \mathcal{F}_N , which are finite dimensional spaces, their trace on \mathcal{F}_N

$$tr_N \mathcal{O} = \sum_{n=0}^N \langle n | \mathcal{O} | n \rangle \quad (\text{A.11})$$

is well-defined and cyclic³. The trace on \mathcal{F} is then defined by the inductive limit

$$tr \mathcal{O} = \lim_{N \rightarrow \infty} \frac{1}{N} tr_N \mathcal{O}. \quad (\text{A.12})$$

It is cyclic and such that

$$tr \mathbb{I} = 1 \quad ; \quad tr \pi^{ev} = tr \bar{\pi}^{ev} = \frac{1}{2} \quad ; \quad tr \pi^{\leq \ell} = 0 \quad \text{and} \quad tr \bar{\pi}^{\leq \ell} = 1. \quad (\text{A.13})$$

In the same way, the permutation operator commutes with $\hat{N}_1 + \hat{N}_2$, so that $tr_{12} P_{12}$ is well-defined and cyclic. However, to define the transfer matrix, we need to use the partial trace $tr_1 P_{12}$ which is ill-defined because P_{12} does not commute with \hat{N}_1 . For instance, it is easy to see

$$tr_{N_1}(P_{12} P_{13}) \neq tr_{N_1}(P_{13} P_{12}) \quad (\text{A.14})$$

where tr_{N_1} is the operator tr_N in the space 1. Equation (A.14) just shows that the partial trace is not cyclic. As a consequence, one cannot prove that transfer matrices with different spectral parameters commute, and the integrability of the model is not guaranteed.

B Twisted version of XX and Hubbard models

The Hubbard models we have constructed depend on a single free parameter U . We present here a construction that allows us to introduce more parameters. In particular, we will obtain an Hermitian Hamiltonian that depends on phases that can be identified with a Aharonov-Bohm phase. Again, we present in detail the construction for XX models, and just sketch the generalization to Hubbard models.

We start with an universal XX model defined on the vector space \mathcal{V} , with a projector π such that $\pi(\mathcal{V}) = \mathcal{W}$.

Definition B.1 *To any projection π , a refinement is a decomposition*

$$\pi = \oplus_{j=1}^d \pi^{(j)} \quad \text{with} \quad \pi^{(j)} \pi^{(j')} = \delta_{j,j'} \pi^{(j)} \quad \forall j, j' = 1, \dots, d$$

For each refinement of π and $\bar{\pi}$, we introduce

$$\mathcal{F}_{12} = \mathbb{I} \otimes \mathbb{I} + \sum_{a=1}^d \sum_{\bar{a}=1}^{\bar{d}} (q_{a\bar{a}} - 1) \pi^{(a)} \otimes \pi^{(\bar{a})}$$

where $q_{a\bar{a}}$ are some non-zero complex numbers. Note that $q_{a\bar{a}} \neq 0$ ensures that \mathcal{F}_{12} is invertible. Its inverse reads

$$\mathcal{F}_{12}^{-1} = \mathbb{I} \otimes \mathbb{I} - \sum_{a=1}^d \sum_{\bar{a}=1}^{\bar{d}} \frac{q_{a\bar{a}} - 1}{q_{a\bar{a}}} \pi^{(a)} \otimes \pi^{(\bar{a})}$$

³In fact it is true for any operator \mathcal{O} obeying $\mathcal{O}(\mathcal{F}_N) \subset \mathcal{F}_N$.

The twisted version of the XX model is defined by the R-matrix

$$\widehat{R}_{12}(\lambda) = \mathcal{F}_{12} R_{12}(\lambda) \mathcal{F}_{21}^{-1}$$

where $R_{12}(\lambda)$ is the matrix (2.4). We remind that it depends on the choice of the projector π .

Property B.2 *The R-matrix $\widehat{R}_{12}(\lambda)$ can be rewritten as*

$$\widehat{R}_{12}(\lambda) = \widehat{\Sigma}_{12} \sin \lambda + \left(\Sigma_{12} + (\mathbb{I} \otimes \mathbb{I} - \Sigma_{12}) \cos \lambda \right) P_{12} \quad (\text{B.1})$$

$$\widehat{\Sigma}_{12} = \mathcal{F}_{12} \Sigma_{12} \mathcal{F}_{21}^{-1} = \sum_{a=1}^d \sum_{\bar{a}=1}^{\bar{d}} \left(q_{a\bar{a}} \pi^{(a)} \otimes \pi^{(\bar{a})} + \frac{1}{q_{a\bar{a}}} \pi^{(\bar{a})} \otimes \pi^{(a)} \right) \quad (\text{B.2})$$

It obeys all the properties stated in theorem 2.1, but the symmetry one.

Proof: The expressions (B.1) and (B.2) follow from direct calculations. Remarking that for any diagonal matrix \mathcal{D}_{12} , we have the property

$$\mathcal{D}_{13} \mathcal{D}_{23} R_{12}(\lambda) = R_{12}(\lambda) \mathcal{D}_{13} \mathcal{D}_{23}$$

the YBE for $\widehat{R}_{12}(\lambda)$ is deduced from the YBE for $R_{12}(\lambda)$. For instance, starting from the l.h.s. and using the notation $\lambda_{ij} = \lambda_i - \lambda_j$:

$$\begin{aligned} \widehat{R}_{12}(\lambda_{12}) \widehat{R}_{13}(\lambda_{13}) \widehat{R}_{23}(\lambda_{23}) &= \mathcal{F}_{12} R_{12}(\lambda_{12}) \mathcal{F}_{21}^{-1} \mathcal{F}_{13} R_{13}(\lambda_{13}) \mathcal{F}_{31}^{-1} \mathcal{F}_{23} R_{23}(\lambda_{23}) \mathcal{F}_{32}^{-1} \\ &= \mathcal{F}_{12} R_{12}(\lambda_{12}) \mathcal{F}_{13} \mathcal{F}_{23} \mathcal{F}_{23}^{-1} \mathcal{F}_{21}^{-1} R_{13}(\lambda_{13}) \mathcal{F}_{31}^{-1} \mathcal{F}_{23} R_{23}(\lambda_{23}) \mathcal{F}_{32}^{-1} \\ &= \mathcal{F}_{12} \mathcal{F}_{13} \mathcal{F}_{23} R_{12}(\lambda_{12}) R_{13}(\lambda_{13}) \mathcal{F}_{31}^{-1} \mathcal{F}_{23}^{-1} \mathcal{F}_{21}^{-1} \mathcal{F}_{23} R_{23}(\lambda_{23}) \mathcal{F}_{32}^{-1} \\ &= \mathcal{F}_{12} \mathcal{F}_{13} \mathcal{F}_{23} R_{12}(\lambda_{12}) R_{13}(\lambda_{13}) R_{23}(\lambda_{23}) \mathcal{F}_{31}^{-1} \mathcal{F}_{21}^{-1} \mathcal{F}_{32}^{-1} \end{aligned}$$

The r.h.s. is treated in the same way.

Regularity is obtained as follows

$$\widehat{R}_{12}(0) = \mathcal{F}_{12} R_{12}(0) \mathcal{F}_{21}^{-1} = \mathcal{F}_{12} P_{12} \mathcal{F}_{21}^{-1} = \mathcal{F}_{12} \mathcal{F}_{12}^{-1} P_{12} = P_{12}$$

Similar calculations lead to the other properties. ■

Property B.2 ensures that models based on twisted R -matrices are also integrable and possess a local Hamiltonian. Indeed, the corresponding Hamiltonian reads

$$\widehat{H}(q_{a\bar{a}}) = \sum_{j=1}^L \widehat{H}_{j,j+1}(q_{a\bar{a}}) \quad \text{with} \quad \widehat{H}_{j,j+1} = P_{j,j+1} \widehat{\Sigma}_{j,j+1}(q_{a\bar{a}}) \quad (\text{B.3})$$

Its hermiticity depends on the parameters $q_{a\bar{a}}$. From the calculation

$$\widehat{H}(q_{a\bar{a}}) = \sum_{j=1}^L P_{j,j+1} \widehat{\Sigma}_{j,j+1}(q_{a\bar{a}}) \quad (\text{B.4})$$

$$\widehat{H}(q_{a\bar{a}})^\dagger = \sum_{j=1}^L \widehat{\Sigma}_{j,j+1}(q_{a\bar{a}}^*) P_{j,j+1} = \sum_{j=1}^L P_{j,j+1} \widehat{\Sigma}_{j+1,j}(q_{a\bar{a}}^*) \quad (\text{B.5})$$

and the identity

$$\widehat{\Sigma}_{j+1,j}(q_{a\bar{a}}) = \widehat{\Sigma}_{j,j+1}\left(\frac{1}{q_{a\bar{a}}}\right)$$

one deduces that \widehat{H} is hermitian when the parameters are phases:

$$q_{a\bar{a}} = e^{i\theta_{a\bar{a}}}, \quad \theta_{a\bar{a}} \in \mathbb{R}.$$

Choosing the parameters $q_{a\bar{a}}$ to be phases, we have in this way a general Hermitian Hamiltonian with Aharonov-Bohm phases on each site of the model.

Twisted Hubbard models: The same construction can be done for Hubbard models, with now

$$\widehat{R}_{12}^{\uparrow\downarrow}(\lambda) = \mathcal{F}_{12}^{\uparrow\downarrow}(\mathbf{q}^{\uparrow}, \mathbf{q}^{\downarrow}) R_{12}^{\uparrow\downarrow}(\lambda) \left(\mathcal{F}_{21}^{\uparrow\downarrow}(\mathbf{q}^{\uparrow}, \mathbf{q}^{\downarrow}) \right)^{-1} \quad \text{with} \quad \mathcal{F}_{12}^{\uparrow\downarrow}(\mathbf{q}^{\uparrow}, \mathbf{q}^{\downarrow}) = \mathcal{F}_{12}^{\uparrow}(\mathbf{q}^{\uparrow}) \mathcal{F}_{12}^{\downarrow}(\mathbf{q}^{\downarrow}).$$

Since \mathcal{F} and C are diagonal, the new R -matrix reads

$$\widehat{R}_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) = \widehat{R}_{12}^{\uparrow}(\lambda_{12}) \widehat{R}_{12}^{\downarrow}(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12}) \widehat{R}_{12}^{\uparrow}(\lambda'_{12}) C_1^{\uparrow} \widehat{R}_{12}^{\downarrow}(\lambda'_{12}) C_1^{\downarrow} \quad (\text{B.6})$$

where $\widehat{R}^{\varepsilon}(\lambda)$, $\varepsilon = \uparrow, \downarrow$, have the form (B.1). It leads to an Hamiltonian

$$\widehat{H} = \sum_{j=1}^L \left(P_{j,j+1}^{\uparrow} \widehat{\Sigma}_{j,j+1}^{\uparrow}(q_{a\bar{a}}^{\uparrow}) + P_{j,j+1}^{\downarrow} \widehat{\Sigma}_{j,j+1}^{\downarrow}(q_{a\bar{a}}^{\downarrow}) + U C_j^{\uparrow} C_j^{\downarrow} \right) \quad (\text{B.7})$$

that is hermitian as soon as the parameters $q_{a\bar{a}}^{\varepsilon}$ are phases.

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